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CONVENIENT CATEGORIES FOR TOPOLOGISTS

H.L. BENTLEY, H. HERRLICH and W.A. ROBERTSON

Abstract: The category Top of R_0-topological spaces is nicely embedded in Grill and Conv, two cartesian closed topological categories of nearness spaces. Grill is a quasi-topos which, although constructible from Top in a natural way, contains the contiguity and proximity spaces as bireflective subcategories. Conv, bicoreflective in Grill, is isomorphic to the category of symmetric convergence spaces.

Key Words: Topological category, cartesian closed category, quasi-topos, merotopic spaces, grill-determined spaces, convergence spaces, products of quotient maps.

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Since the category Top of topological spaces and continuous maps fails to have some desirable properties (for example products don't commute with quotients and there is in general no natural mapping space topology; that is, Top is not cartesian closed), there have been various attempts to replace Top by more convenient categories. Unfortunately most of these categories suffer from other deficiencies. Some are too small (e.g. sequential spaces [28]), some too large (e.g. quasi-topological spaces [23]). Others like the category of compactly generated spaces [25], a large subcategory of Top, and the category of epitopological spaces [1], a small supercategory of Top, have not been described directly by suitable axioms.

In this paper the authors propose in the realm of nearness structures [10, 11] categories Grill and Conv, which satisfy the criteria for convenient topological categories set up by N.E. Steenrod [24] and are free from the abovementioned deficiencies. Furthermore, Conv is intimately related to such cartesian closed categories as the convergence spaces [17], the limit spaces [8, 18] and the pseudotopological spaces [6], yet its spaces have structures which are less point-bound. Grill contains Conv, and also some important topological categories not embeddable in the convergence spaces. Categories equivalent to Grill and Conv were introduced as early as 1965, but in a different context, by M. Katětov [15, 16].

§ 1. The category P-Near

1.1. Definition. Let X be a set. For $\mathfrak{A} \subset PX = \{A \mid A \subset X\}$, let stack $\mathfrak{A}$ = $\mathrm{stack}_X \mathfrak{A} = \{B \subset X \mid A \subset B$ for some $A \in \mathfrak{A}\}$. If $\mathfrak{B} \subset PX$, $\mathfrak{B}$ corefines $\mathfrak{A}$ iff $\mathfrak{B} \subset$ stack $\mathfrak{A}$.

1.2. Definitions. A subset ξ of PPX is called a prenearness structure on X if it satisfies the following conditions:

(N1) If $\mathfrak{B}$ corefines $\mathfrak{A}$ and $\mathfrak{A} \in \xi$, then $\mathfrak{B} \in \xi$.

(N2) If $\bigcap \mathfrak{A} \neq \emptyset$, then $\mathfrak{A} \in \xi$.

(N3) $\{\emptyset\} \notin \xi$; $\emptyset \in \xi$.

If ξ is a prenearness structure on X, then (X,ξ) is called a prenearness space. A map $f: (X,\xi) \longrightarrow (Y,\eta)$ between prenearness spaces is called nearness preserving if $\mathcal{A} \in \xi$ implies $f\mathcal{A} = \{fA \mid A \in \mathcal{A}\} \in \eta$. The category of prenearness spaces and nearness preserving maps is denoted by P-Near.

1.3. Remarks. The category P-Near was introduced by H. Herrlich [11]. It is a properly fibred topological category [11] and as such has a number of pleasant properties (c.f., for example, P. Antoine [1], H. Herrlich [11], R.E. Hoffman [13], M. Hušek [14] and O. Wyler [26, 27]). In particular, P-Near is complete, cocomplete, wellpowered and cowellpowered, embeddings = extremal monomorphisms = regular monomorphisms, quotient maps = extremal epimorphisms = regular epimorphisms, any object with non-empty underlying set is a separator, and the forgetful functor P-Near $\longrightarrow$ Set has a full and faithful left adjoint and a full and faithful right adjoint. Embeddings, quotient maps, limits and colimits are characterized as follows:

(a) A map $f: (X,\xi) \longrightarrow (Y,\eta)$ between two prenearness spaces is an embedding in P-Near iff f is injective and $\xi = \{\mathcal{A} \subset PX \mid f\mathcal{A} \in \eta\}$. If f is an embedding, (X,ξ) is called a (prenearness) subspace of (Y,η). f is a quotient map in P-near iff f is onto and $\eta = \{\mathcal{B} \subset PY \mid f^{-1}\mathcal{B} = \{f^{-1}B \mid B \in \mathcal{B}\} \in \xi\}$. If f is a quotient map, (Y,η) is called a (prenearness) quotient of (X,ξ).

(b) A non-empty source $(p_i: (X,\xi) \longrightarrow (X_i,\xi_i))_{i\in I}$ in P-Near is a limit of some diagram in P-Near iff $(p_i: X \longrightarrow X_i)_{i\in I}$ is a limit of the underlying diagram in Set and

$\xi = \{\mathcal{A} \subset PX \mid p_i \mathcal{A} \in \xi_i \text{ for each } i \in I\}$.

(c) A non-empty sink $(g_i: (X_i, \xi_i) \longrightarrow (X, \xi))_{i\in I}$ in P-Near is a colimit of some diagram in P-Near iff $(g_i: X_i \longrightarrow X)_{i\in I}$ is a colimit of the underlying diagram in Set and

$\xi = \{\mathcal{A} \subset PX \mid g_i^{-1} \mathcal{A} \in \xi_i \text{ for some } i \in I\}$.

1.4. Theorem. In P-Near any product of quotient maps is a quotient map.

Proof. Let $(f_i: (X_i, \xi_i) \longrightarrow (Y_i, \eta_i))_{i\ I}$ be a non-empty family of quotient maps in P-Near, let $(p_i: (P, \xi) \longrightarrow (X_i, \xi_i))_{i\in I}$ and $(q_i: (Q, \eta) \longrightarrow (Y_i, \eta_i))_{i\in I}$ be products in P-Near, and let $f = \Pi f_i: (P, \xi) \longrightarrow (Q, \eta)$. Since all $f_i: X_i \longrightarrow Y_i$ are onto, so is $f: P \longrightarrow Q$. If $\mathcal{B} \in \eta$, then $q_i \mathcal{B} \in \eta_i$ for each $i \in I$. Hence $p_i f^{-1} \mathcal{B} = f_i^{-1} q_i \mathcal{B} \in \xi_i$ for each $i \in I$. Consequently $f^{-1}\mathcal{B} \in \xi$, which implies $f: (P, \xi) \longrightarrow (Q, \eta)$ is a quotient map in P-Near.

1.5. Remark. In P-Near products don't commute with coproducts, as the following example shows. From the characterization of cartesian closedness for topological categories in [12], we conclude that P-Near is not cartesian closed. In § 2 and § 3 we will turn our attention to nicely embedded subcategories of P-Near which have the desired properties.

1.6. Example. If $X = \{1,2\}$, $Y_1 = \{1\}$, $Y_2 = \{2\}$, $\xi = \{\mathcal{A} \subset PX \mid \cap \mathcal{A} \neq \phi\}$ and $\xi_i = \{\mathcal{A} \subset PY_i \mid \cap \mathcal{A} \neq \phi\}$ for $i = 1,2$, then $\underline{X} = (X, \xi)$, $\underline{Y}_1 = (Y_1, \eta_1)$ and $\underline{Y}_2 = (Y_2, \eta_2)$ are prenearness spaces. But $\underline{X} \times \coprod \underline{Y}_i \not\approx \coprod (\underline{X} \times \underline{Y}_i)$ since $\mathcal{A} = \{\{(1,1)\}, \{(1,2), (2,1)\}\}$ is a near collection in $\underline{X} \times \coprod \underline{Y}_i$ but not in $\coprod (\underline{X} \times \underline{Y}_i)$.

1.7. Remarks. (1) Several topological categories can be nicely embedded as subcategories of P-Near. (By subcategory, we mean full, isomorphism closed subcategory.) Among these are

(a) The category S-Near of seminearness spaces whose objects are those prenearness spaces (X, ξ) satisfying

(N4) $\mathcal{A} \vee \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}$ and $B \in \mathcal{B}\} \in \xi$ implies $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$.

S-Near is a bicoreflective subcategory of P-Near [11] which is isomorphic to the category of merotopic spaces introduced in a fascinating but little-known paper by M. Katětov [15].

(b) The category Near of nearness spaces whose objects are those seminearness spaces (X, ξ) satisfying

(N5) $\mathrm{cl}_\xi \mathcal{A} = \{\mathrm{cl}_\xi A \mid A \in \mathcal{A}\} \in \xi$ implies $\mathcal{A} \in \xi$ (where $\mathrm{cl}_\xi A = \{x \in X \mid \{\{x\}, A\} \in \xi\}$).

Near is a bireflective subcategory of S-Near [11].

(c) The category whose objects are those nearness spaces (X, ξ) such that

$$\mathcal{A} \in \xi \quad \text{iff} \quad \bigcap \mathrm{cl}_\xi \mathcal{A} \neq \emptyset .$$

This category is isomorphic to the category of topological R_0 spaces (topological spaces (X,cl) satisfying the condition that $x \in \mathrm{cl}\{y\}$ implies $y \in \mathrm{cl}\{x\}$) and will be denoted by Top. The isomorphism identifies an R_0 space (X,cl) with the nearness space (X, ξ), where $\xi = \{\mathcal{A} \subset PX \mid \bigcap \mathrm{cl}\,\mathcal{A} \neq \emptyset\}$. Top is bicoreflective in Near [10].

(d) The categories of uniform spaces, contiguity spaces and proximity spaces, each of which can be embedded bireflectively in Near [10].

(2) None of the categories mentioned in (a) - (d) are cartesian closed. This is well-known for the category of topological spaces and the subcategory of R_0 spaces. The following example shows it for S-Near, Near, and the category Unif of uniform spaces. A slight modification of the example shows this for contiguity and proximity spaces.

1.8. Example. Let $\underline{X}$ be the set $[0,1]$ with nearness structure ξ induced by the usual topology on $[0,1]$ (see 1.7(c)). For each $n \in N$, let $\underline{Y}_n$ be the (unique) nearness space with underlying set $\{n\}$. In S-Near, each of $\underline{X} \times \coprod \underline{Y}_n$ and $\coprod(\underline{X} \times \underline{Y}_n)$ has underlying set $[0,1] \times N$. If $A = \{0\} \times N$ and $B = \{(\frac{1}{n}, n) \mid n \in N\}$, then $\{A,B\}$ is a near collection in $\underline{X} \times \coprod \underline{Y}_n$, but not in $\coprod(\underline{X} \times \underline{Y}_n)$. Since Near and Unif are closed under the formation of products and coproducts in S-Near, we obtain $\underline{X} \times \coprod \underline{Y}_n \not\approx \coprod(\underline{X} \times \underline{Y}_n)$ in any of these categories.

§ 2. The category Grill

2.1. <u>Definitions</u>. (1) $\mathcal{G} \subset PX$ is called a grill in X iff $\phi \notin \mathcal{G}$ and the following condition is satisfied:

For any $A,B \subset X$, $A \cup B \in \mathcal{G}$ iff $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

(2) If (X, ξ) is a prenearness space then $\mathcal{G}$ is called a ξ-grill iff $\mathcal{G}$ is a grill and $\mathcal{G} \in \xi$.

2.2. Remarks. The concept of grill was introduced by G. Choquet [5]. The notion is dual to that of filter. The ultrafilters are the minimal grills; equivalently, they are the filters which are grills. Grills are precisely the unions of ultrafilters, and the union of grills is a grill.

2.3. <u>Definitions</u>. A prenearness space (X, ξ) is called grill-determined iff each $\mathcal{A} \in \xi$ is contained in some ξ-grill $\mathcal{G}$. The subcategory of P-Near whose objects are the grill-determined spaces is denoted by Grill.

2.4. Remarks. (1) If X is any set and G a collection of grills in X satisfying the condition that for each $x \in X$ there exists a grill $\mathcal{G}$ in G with $\{x\} \in \mathcal{G}$, then $\xi = \{\mathcal{A} \subset PX \mid \mathcal{A} \subset \mathcal{G}$ for some $\mathcal{G} \in G\}$ is a grill-determined prenearness structure. We say the collection G determines ξ.

(2) The category Grill was introduced by W.A. Robertson [22]. It is isomorphic to the category of filter merotopic spaces introduced by M. Katětov [15]. It is still big enough to contain the categories of contiguity spaces, proximity spaces, and topological R_0 spaces.

(3) If (X, ξ) is a prenearness space, then the set ξ_g of all $\mathcal{A} \in \xi$ which are contained in some ξ-grill is a grill-determined prenearness structure on X, and the map $1_X: (X, \xi_g) \longrightarrow (X, \xi)$ is a Grill-coreflection of (X, ξ).

As a bicoreflective subcategory of P-Near, Grill is a properly fibred topological category with final structures (and hence colimits) formed as in P-Near and initial structures (and hence limits) formed by forming them first in P-Near and applying the Grill-coreflector. In particular, any prenearness subspace of a grill-determined space is grill-determined. Products are described explicitly in 2.6.

2.5. Lemma. Let $(X_i)_{i\in I}$ be a non-empty family of sets, $(p_j\colon \prod X_i \to X_j)_{j\in I}$ the cartesian product and $\mathcal{G}_i$ a grill in X_i for each $i\in I$. Then there exists a largest grill in $\prod X_i$, denoted by $\otimes\, \mathcal{G}_i$, with the property that $p_j(\otimes\, \mathcal{G}_i) \subset \mathcal{G}_j$ for each $j\in I$.

(a) The following statements are equivalent for $G \subset \prod X_i$:

(i) $G \in \otimes\, \mathcal{G}_i$.

(ii) If $\mathcal{B}$ is a finite cover of G, then there exists $B\in\mathcal{B}$ with $p_i B \in \mathcal{G}_i$ for each $i\in I$.

(iii) If $G \subset \bigcup\{p_j^{-1}A_j \mid j\in J\}$, where $J\subset I$ is finite, then there exists $j_o\in J$ with $A_{j_o} \in \mathcal{G}_{j_o}$.

(b) In the case all X_i are non-empty we have $p_j(\otimes\, \mathcal{G}_i) = \mathcal{G}_j$ for each $j\in I$.

Proof. Conditions (ii) and (iii) may be readily seen to define grills with i^{th} projection a subset of $\mathcal{G}_i$ for each $i\in I$. Conditiom (ii) defines the largest such grill. For suppose $\mathcal{A}$ is a grill in $\prod X_i$ with $p_i\mathcal{A} \subset \mathcal{G}_i$ for each $i\in I$, and $A\in\mathcal{A}$. If $\mathcal{B}$ is a finite cover of A, then $B\in\mathcal{A}$ for some $B\in\mathcal{B}$. That means $p_iB \in \mathcal{G}_i$ for each $i\in I$. Clearly (ii) implies (iii) and all the statements are equivalent.

2.6. Remark. If (X_i, ξ_i) is a grill-determined space for each $i\in I$, the collection $\{\otimes\, \mathcal{G}_i \mid (\mathcal{G}_i)_{i\in I}$ is a family of ξ_i-grills $\mathcal{G}_i\}$ determines the product structure on $\prod X_i$ in Grill.

2.7. Remark. If $\mathcal{G}$ is a grill in X and $f\colon X \to Y$ is any Set-map, stack $f\mathcal{G}$ is a grill in Y. Notice that if $\mathcal{H}$ is a grill in Y, $f^{-1}\mathcal{H}$ need not be a grill in X.

2.8. Lemma. Let $(f_i: X_i \longrightarrow Y_i)_{i\in I}$ be a non-empty family of Set-maps, $(p_i: P \longrightarrow X_i)_{i\in I}$ and $(q_i: Q \longrightarrow Y_i)_{i\in I}$ products in Set, and $f = \prod f_i: P \longrightarrow Q$. Let $\mathcal{G}_i$ be a grill in X_i for each $i \in I$. Then

(a) $\otimes$ stack $f_i\ \mathcal{G}_i$ = stack $f(\otimes\ \mathcal{G}_i)$.

(b) If $K = \{i \in I \mid f_i X_i \neq Y_i\}$ is finite and $\mathcal{H}_i$ is a grill in Y_i with $f_i^{-1}\mathcal{H}_i \subset \mathcal{G}_i$ for each $i \in I$, then $f^{-1}(\otimes\ \mathcal{H}_i) \subset \subset \otimes\ \mathcal{G}_i$.

Proof. (a) is straightforward.

(b) Assume $f^{-1}H \notin \otimes\ \mathcal{G}_i$ for some $H \in \otimes\ \mathcal{H}_i$. Lemma 2.5 implies that there exists finite $J \subset I$ and $A_j \notin \mathcal{G}_j$ for each $j \in J$ so that $f^{-1}H \subset \bigcup\{p_j^{-1}A_j \mid j \in J\}$. If B_j is the largest subset of A_j with $f_j^{-1}f_jB_j = B_j$, then $B_j \notin \mathcal{G}_j$ and $f^{-1}H \subset \bigcup\{p_j^{-1}B_j \mid j \in J\}$, since for each $y \in H$, $f^{-1}(y) \subset \subset \bigcup\{p_j^{-1}A_j \mid j \in J\}$ implies that $\prod(f_i^{-1}q_i(y)) = f^{-1}(y) \subset \subset p_j^{-1}A_j$ for some $j \in J$, and hence $f^{-1}(y) \subset p_j^{-1}B_j$ for that j. So $ff^{-1}H \subset \bigcup\{fp_j^{-1}B_j \mid j \in J\}$ and $H \subset \bigcup\{fp_j^{-1}B_j \mid j \in J\} \cup \cup \bigcup\{q_k^{-1}(Y_k \setminus f_kX_k) \mid k \in K\}$. Then some member of this finite cover must be in $\otimes\ \mathcal{H}_i$. But $q_k^{-1}(Y_k \setminus f_kX_k) \notin \otimes\ \mathcal{H}_i$ for any $k \in K$ since $f_k^{-1}q_k(q_k^{-1}(Y_k \setminus f_kX_k)) = \phi \notin \mathcal{G}_k$, and $fp_j^{-1}B_j \notin \notin \otimes\ \mathcal{H}_i$ for any $j \in J$ since $f_j^{-1}q_j(fp_j^{-1}B_j) = f_j^{-1}f_jB_j = B_j \notin \notin \mathcal{G}_j$ (contradiction).

2.9. Remark. The condition that K be finite is necessary. For suppose K is infinite, and let $\mathcal{G}_i = PX_i \setminus \{\phi\}$, $\mathcal{H}_i = \{B \subset Y_i \mid B \cap f_iX_i \neq \phi\}$ for each $i \in I$. Then $f_i^{-1}\mathcal{H}_i \subset \subset \mathcal{G}_i$ for each $i \in I$, but $Q \setminus fP \in \otimes\ \mathcal{H}_i$ and $f^{-1}(Q \setminus fP) =$

$= \phi \notin \otimes \mathcal{G}_i.$

2.10. Theorem. In Grill any product of quotient maps is a quotient map.

Proof. Let $(f_i\colon (X_i, \xi_i) \longrightarrow (Y_i, \eta_i))_{i\in I}$ be a non-empty family of quotient maps in Grill, let $(p_i\colon (P, \xi) \longrightarrow$ $\longrightarrow (X_i, \xi_i))_{i\in I}$ and $(q_i\colon (Q, \eta) \longrightarrow (Y_i, \eta_i))_{i\in I}$ be products in Grill, and let $f = \prod f_i\colon (P, \xi) \longrightarrow (Q, \eta)$. Since all $f_i\colon$ $\colon X_i \longrightarrow Y_i$ are onto, so is $f\colon P \longrightarrow Q$. If $\mathcal{B} \in \eta$, then $\mathcal{B} \subset \mathcal{H}$ for some η-grill $\mathcal{H}$. Then $q_i \mathcal{H} \in \eta_i$ and hence $f_i^{-1} q_i \mathcal{H} \in \xi_i$ for each $i \in I$. Consequently for each $i \in I$ there is a ξ_i-grill $\mathcal{G}_i$ with $f_i^{-1} q_i \mathcal{H} \subset \mathcal{G}_i$. $\otimes \mathcal{G}_i$ is a ξ-grill, and 2.8(b) implies that $f^{-1}(\otimes q_i \mathcal{H}_i) \subset \otimes \mathcal{G}_i$. But $f^{-1}\mathcal{B} \subset f^{-1}\mathcal{H} \subset f^{-1}(\otimes q_i \mathcal{H}_i)$; so $f^{-1}\mathcal{B} \in \xi$.

2.11. Theorem. In Grill finite products commute with direct limits.

Proof. Consider a commutative diagram in Grill

$$\begin{array}{ccc} (P_k, \xi_k) & \xrightarrow{\;f_k\;} & (P, \xi) \\ {\scriptstyle p_{ik}}\downarrow & & \downarrow{\scriptstyle p_i} \\ (X_{ik}, \xi_{ik}) & \xrightarrow[\;f_{ik}\;]{} & (X_i, \xi_i) \end{array}$$

$i \in I$, I finite, non-empty. $k \in K$, $(K, \leq)$ a directed set.

where the columns (for fixed k) represent finite products and the bottom rows (for fixed i) represent direct limits in Grill. To see that the top row represents a direct limit in Grill, observe that it represents a direct limit on the Set level. It remains to show that for any ξ-grill $\mathcal{A}$ there exists $k \in K$ with $f_k^{-1}\mathcal{A} \in \xi_k$. For each $i \in I$, $p_i \mathcal{A} \in \xi_i$. Consequently for each $i \in I$ there exists $k(i) \in K$ such that $f_{ik(i)}^{-1} p_i \mathcal{A} \in \xi_{ik(i)}$. Since I is finite and $(K, \leq)$ is directed, there exists $k_o \in K$ with $k(i) \leq k_o$ for each $i \in I$. Therefore, $f_{ik_o}^{-1} p_i \mathcal{A} \in \xi_{ik_o}$ for each $i \in I$. 2.8(b) implies $f_{k_o}^{-1} \mathcal{A} \in \xi_{k_o}$.

2.12. Theorem. In Grill $\coprod(\underline{X}_i \times \underline{Y}_j) \approx \coprod \underline{X}_i \times \coprod \underline{Y}_j$.

Proof. First consider a commutative diagram in Grill

$$\begin{array}{ccc}
\underline{Y}_i = (Y_i, \eta_i) & \xrightarrow{\quad m_i \quad} & (Y, \eta) = \coprod \underline{Y}_i \\
p_i \uparrow & & \uparrow p_Y \\
\underline{X} \times \underline{Y}_i = (X \times Y_i, \gamma_i) & \xrightarrow{\quad 1_X \times m_i \quad} & (X \times Y, \gamma) = \underline{X} \times \coprod \underline{Y}_i \\
p_X \downarrow & & \downarrow p_X \\
\underline{X} = (X, \xi) & \xrightarrow[\quad 1_X \quad]{} & (X, \xi) = \underline{X}
\end{array}$$

in which the top row represents a non-empty coproduct and the columns products in Grill. To see that the middle row represents a coproduct, observe that $(1_X \times m_i : X \times Y_i \longrightarrow X \times Y)_{i \in I}$ represents a coproduct in Set. Hence it remains to show that for $\mathcal{A} \in \gamma$, $(1_X \times m_j)^{-1} \mathcal{A} \in \gamma_j$ for some $j \in I$. Now $\mathcal{A}$ is contained in a γ-grill $\mathcal{G}$. $p_X \mathcal{G} \in \xi$ and $p_Y \mathcal{G} \in \eta$ are grills. For some $j \in I$, $m_j^{-1} p_Y \mathcal{G} \in \eta_j$; that is, $m_j^{-1} p_Y \mathcal{G} \subset \mathcal{H}$ for some η_j-grill $\mathcal{H}$. But 2.8(b) implies that $(1_X \times m_j)^{-1} \mathcal{G} \subset p_X \mathcal{G} \otimes \mathcal{H} \in \gamma_j$. Therefore $(1_X \times m_j)^{-1} \mathcal{A} \in \gamma_j$. This proves $\underline{X} \times \coprod \underline{Y}_i = \coprod (\underline{X} \times \underline{Y}_i)$. Apply this formula twice to get the desired result.

2.13. Theorem. Grill is a quasi-topos in the sense of

Penon, i.e.

(1) Grill is cartesian closed.

(2) In Grill, for each $\underline{X}$ there exists an embedding $\underline{X} \to \underline{X}'$ with the property that for each embedding $\underline{Y} \to \underline{Y}'$ and each morphism $\underline{Y} \to \underline{X}$ there exists a unique morphism $\underline{Y}' \to \underline{X}'$ making the diagram

$$\begin{array}{ccc} \underline{Y} & \longrightarrow & \underline{X} \\ \downarrow & & \downarrow \\ \underline{Y}' & \longrightarrow & \underline{X}' \end{array}$$

a pullback.

Proof. (1) Immediate from 2.10, 2.12 and the characterization of cartesian closed topological categories in [12].

(2) For a grill determined space (X, ξ), let x' be a new point not in X, $X' = X \cup \{x'\}$, and $\xi' = \{\mathcal{A} \subset PX' \mid \{A \in \mathcal{A} \mid A \subset X\} \in \xi\}$. Then $(X, \xi) \to (X', \xi')$ is the desired embedding.

2.14. Remarks. (1) Grill is a cartesian closed topological category satisfying Steenrod's criteria for convenient topological categories. Being a quasi-topos, Grill has several other pleasant properties, e.g. colimits are universal, i.e. preserved by pullbacks.

(2) Theorem 2.13 says that if $\underline{X}$ and $\underline{Y}$ are grill-determined spaces Hom($\underline{X}$,$\underline{Y}$), the set of all nearness preserving maps from $\underline{X}$ to $\underline{Y}$, has a natural grill-determined structure Θ, called the power structure of Hom($\underline{X}$,$\underline{Y}$). $\underline{Y}^{\underline{X}} = (\mathrm{Hom}(\underline{X},\underline{Y}), \Theta)$ will be called a power. "Natural" means that

the evaluation map $e_{X,Y}: \underline{X} \times \underline{Y}^{\underline{X}} \longrightarrow \underline{Y}$ defined by $e_{X,Y}(x,g) = g(x)$ is nearness preserving, and for any grill-determined space $\underline{Z}$ and nearness preserving map $f: \underline{X} \times \underline{Z} \longrightarrow \underline{Y}$ in Grill there exists a unique nearness preserving map $\overline{f}: \underline{Z} \longrightarrow \underline{Y}^{\underline{X}}$ so that $f = e_{X,Y} \circ (1_X \times \overline{f})$.

The next theorem explicitly describes the power structure θ , thus providing an alternate proof for Theorem 2.13.

2.15. Theorem. If $\underline{X} = (X, \xi)$ and $\underline{Y} = (Y, \eta)$ are grill-determined spaces, the power structure on Hom($\underline{X}$,$\underline{Y}$) is the structure θ determined by those grills $\mathcal{G}$ in Hom($\underline{X}$,$\underline{Y}$) for which $e_{X,Y}(\mathcal{A} \otimes \mathcal{G}) \in \eta$ for every ξ-grill $\mathcal{A}$.

Proof. Let $g \in$ Hom($\underline{X}$,$\underline{Y}$). Then $\dot{g} = \{B \subset \mathrm{Hom}(\underline{X},\underline{Y}) \mid g \in B\}$ is a grill and $\{g\} \in \dot{g}$. If $\mathcal{A}$ is a ξ-grill, $e_{X,Y}(\mathcal{A} \otimes \dot{g}) = \mathrm{stack}\; g\mathcal{A}$. In view of Remark 2.4(1), is a grill-determined structure.

The evaluation map is clearly nearnes preserving. Suppose $\underline{Z} = (Z, \zeta)$ is a grill-determined space and $f: \underline{X} \times \underline{Z} \longrightarrow \underline{Y}$ a nearness preserving map. If we define $\overline{f}: \underline{Z} \longrightarrow \underline{Y}^{\underline{X}}$ by $\overline{f}(z) = g_Z$, where $g_Z(x) = f(x,z)$, then $f = e_{X,Y} \circ (1_X \times \overline{f})$. To show that $\overline{f}$ is nearness preserving, choose $\mathcal{B} \in \zeta$. Then $\mathcal{B} \subset \mathcal{H}$ for some ζ-grill $\mathcal{H}$. If $\mathcal{A}$ is a ξ-grill, $e_{X,Y}(\mathcal{A} \otimes \mathrm{stack}\; \overline{f}\mathcal{H}) = e_{X,Y}\, \mathrm{stack}\, (1_X \times \overline{f})(\mathcal{A} \otimes \mathcal{H}) = \mathrm{stack}\; e_{X,Y}(1_X \times \overline{f})(\mathcal{A} \otimes \mathcal{H}) = \mathrm{stack}\; f(\mathcal{A} \otimes \mathcal{H})$, a member of η . Hence $\mathrm{stack}\; \overline{f}\mathcal{H} \in \theta$, and since $\overline{f}\mathcal{B} \subset \mathrm{stack}\; \overline{f}\mathcal{H}$ $\overline{f}\mathcal{B} \in \theta$.

§ 3. The category Conv

3.1. Definitions. A grill-determined prenearness space

(X, ξ) is called a convergence space iff for any non-empty member $\mathfrak{A}$ of ξ there exists $x \in X$ with $\mathfrak{A} \cup \{\{x\}\} \in \xi$. The subcategory of Grill whose objects are the convergence spaces is denoted by Conv.

3.2. Remarks. (1) The category Conv was introduced by W.A. Robertson [22]. It is a bicoreflective subcategory of Grill with coreflector described as follows: For any grill-determined space (X, ξ) the set $\xi_C = \{\mathfrak{A} \subset PX \mid$ there exists $x \in X$ with $\mathfrak{A} \cup \{\{x\}\} \in \xi\} \cup \{\phi\}$ is a structure making (X, ξ_C) a convergence space and $1_X: (X, \xi_C) \longrightarrow (X, \xi)$ a Conv-coreflection of (X, ξ).

As a bicoreflective subcategory of Grill, Conv is a properly fibred topological category with final structures (and hence colimits) formed as in Grill (and hence as in P-Near), and initial structures (and hence limits) formed by forming them in Grill and applying the Conv-coreflector.

(2) Conv is isomorphic to the category of localized filter merotopic spaces introduced by M. Katětov [15]. It is also isomorphic to the category of convergence spaces in the sense of D. Kent [17] (which contains the category of limits spaces, introduced independently by H.J. Kowalsky [18] and H.R. Fischer [8], the category of pseudotopological spaces, introduced by G. Choquet [6], and the category of topological spaces as subcategories), provided we assume (as we shall) the following symmetry condition: If $\mathcal{F}$ is a convergent filter with $x \in \bigcap \mathcal{F}$, then $\mathcal{F}$ converges to x. All T_1 convergence spaces satisfy the condition, and for topological spa-

ces it is precisely the R_0 axiom. The embedding of topological spaces is precisely the embedding described in 1.7(c). In fact Top is a bireflective subcategory of Conv, and the Conv-reflector is the restriction of the Near-reflector on S-Near. Hence Top is the intersection of the categories Conv and Near.

3.3. <u>Theorem.</u> Conv is closed under the formation of products in Grill.

Proof. Let $((X_i, \xi_i))_{i\in I}$ be a family of convergence spaces and $(p_i: (P, \xi) \longrightarrow (X_i, \xi_i))_{i\in I}$ the product of this family in Grill. If $\mathfrak{A} \in \xi$ is non-empty, $\mathfrak{A}$ is contained in some ξ-grill $\mathfrak{G}$. For each $i \in I$, $p_i \mathfrak{G} \in \xi_i$. Thus for each $i \in I$ there exists $x_i \in X_i$ with $p_i \mathfrak{G} \cup \{\{x_i\}\} \in \xi_i$. If x is the element of P with $p_i(x) = x_i$ for each $i \in I$, $\mathfrak{G} \cup \dot{x}$ (where $\dot{x} = \{A \subset X \mid x \in A\}$) is a ξ-grill. Hence $\mathfrak{A} \cup \{\{x\}\} \in \xi$.

Immediate consequences are

3.4. <u>Theorem.</u> In Conv the following hold:

(a) Any product of quotient maps is a quotient map.

(b) Finite products commute with direct limits.

(c) $\coprod (\underline{X}_i \times \underline{Y}_j) \approx \coprod \underline{X}_i \times \coprod \underline{Y}_j$.

(d) Conv is cartesian closed.

3.5. <u>Theorem.</u> Powers are formed in Conv by forming them in Grill and applying the Conv-coreflector.

Proof. If $\underline{X}$ and $\underline{Y}$ are convergence spaces, denote the Conv-coreflection of $\underline{Y}^{\underline{X}}$ by $\underline{Y}^{\underline{X}}_C$. Since products in Conv are formed as in Grill, $e_{X,Y}: \underline{X} \times \underline{Y}^{\underline{X}}_C \longrightarrow \underline{Y}$ is a nearness preserving map. Let $\underline{Z}$ be a convergence space and $f: \underline{X} \times \underline{Z} \longrightarrow \underline{Y}$ a nearness preserving map. Then there exists a nearness preserving map $\bar{f}: \underline{Z} \to \underline{Y}^{\underline{X}}$ such that $f = e_{X,Y}(1_X \times \bar{f})$. Applying the Conv-coreflector, we conclude $\bar{f}: \underline{Z} \to \underline{Y}^{\underline{X}}_C$ is nearness preserving.

That powers in Conv are in general distinct from those formed in Grill is shown by the following example.

3.6. Example. Let $\underline{X}$ be the rationals with prenearness structure induced by the usual topology and $\underline{Y}$ the rationals with prenearness structure induced by the topology with subbase for closed sets consisting of the set $A = \{\frac{1}{n} \mid n \in N\}$ together with all sets closed in the usual topology. The grill $\mathcal{G} = \{G \subset Hom(\underline{X},\underline{Y}) \mid$ for each $\varepsilon > 0$ there exists $f \in G$ with $fX \cap A = \phi$ and $|x - f(x)| < \varepsilon$ for every $x \in X\}$ is in the power structure in Grill but not in Conv.

3.7. Remark. Conv is not the smallest cartesian closed category containing Top and inducing the usual topological limits. We mention here three subcategories of Conv. All are cartesian closed categories with powers formed as in Conv, and all are bireflective in Conv.

(a) The category Lim of limit spaces has as objects those convergence spaces (X, ξ) satisfying the condition

$$\mathcal{A} \cup \{\{x\}\} \in \xi \text{ and } \mathcal{B} \cup \{\{x\}\} \in \xi \text{ implies } \mathcal{A} \cup \cup \mathcal{B} \in \xi .$$

Lim, introduced in [22], is isomorphic to the category of limit spaces in the sense of H.J. Kowalsky [18] and H.R. Fischer [8] (see 3.2 (2)).

(b) The category PSTop of pseudotopological spaces has a objects those convergence spaces (X, ξ) satisfying the following condition:

If $\mathcal{G}$ is a grill, and there exists $x \in X$ such that for every ultrafilter $\mathcal{U} \subset \mathcal{G}$, $\mathcal{U} \cup \{\{x\}\} \in \xi$, then $\mathcal{G} \in \xi$.

PsTop, introduced in [22], is isomorphic to the category of pseudotopological spaces in the sense of G. Choquet [6] (see 3.2 (2)).

(c) The category EpiTop of epitopological spaces has as objects those convergence spaces (X, ξ) for which ξ is initial in Conv with respect to $(X, (f_i), ((\underline{B}_i^{\underline{A}_i})_C))_{i \in I}$, where $\underline{A}_i$ and $\underline{B}_i$ are topological spaces and $f_i : X \rightarrow \mathrm{Hom}(\underline{A}_i, \underline{B}_i)$ is a Set-map for each $i \in I$ (c.f. P. Antoine [1] and A. Machado [19]).

§ 4. Constructing Grill from Top

We have noted that the category Grill has among its nicely embedded subcategories not only Top, but also the categories Cont of contiguity spaces and Prox of proximity spaces, neither of which is embeddable in Top or even Conv. On the other hand, every grill-determined space arises naturally from elementary constructions on topological spaces in P-Near.

4.1. Theorem. The convergence spaces are precisely the prenearness quotients of the topological spaces. Thus Conv is the coreflective hull in P-Near of Top.

Proof. If (X, ξ) is a convergence space, there exists a family $(\mathcal{G}_i)_{i \in I}$ of ξ-grills which determines ξ and such that for each $i \in I$ there exists $x_i \in X$ with $\{x_i\} \in \in \mathcal{G}_i$. For each $i \in I$, let $\xi_i = \{\mathcal{A} \subset PX \mid \mathcal{A} \subset \mathcal{G}_i$ or $\bigcap \mathcal{A} \neq \phi\}$. Then (X, ξ_i) is in Top and $1_X: (X, \xi_i) \longrightarrow \longrightarrow (X, \xi)$ is a nearness preserving map. Let (Y, η) be the coproduct of $((X, \xi_i))_{i \in I}$. Since Top is closed under the formation of coproducts in P-Near, (Y, η) is in Top. The map $f: (Y, \eta) \longrightarrow (X, \xi)$ induced by $(1_X: (X, \xi_i) \longrightarrow \longrightarrow (X, \xi))_{i \in I}$ is a quotient map.

The converse is immediate from Remark 3.2 (1).

4.2. Remark. Since each of the categories Conv, Grill and S-Near is bicoreflective in P-Near, we conclude that Conv is the coreflective hull of Top in each of these categories, in particular in S-Near.

4.3. Theorem. The grill-determined spaces are precisely the prenearness subspace of the convergence spaces.

Proof. If (X, ξ) is a grill-determined space, let $X' = \{\mathcal{G} \subset PX \mid \mathcal{G}$ is a ξ-grill$\}$ and define a map $i: X \longrightarrow \longrightarrow X'$ by $i(x) = \dot{x}$. The structure ξ on X induces a structure ξ' on X' as follows: Define

$\xi' = \{\Omega \subset PX' \mid \bigcup \{\bigcap \omega \mid \omega \in \Omega\} \in \xi\}$. (X', ξ') is a convergence space and $i: (X, \xi) \longrightarrow (X', \xi')$ is an embedding.

The converse follows in view of Remark 2.4 (3).

4.4. Remarks. (1) 4.3 and 3.3 imply that Grill is the epireflective hull of Conv in Grill. As has been shown by M. Shayegan Hastings [9] the epireflective hull of Grill (and thus Conv) in S-Near is S-Near.

(2) 4.1 and 4.3 imply that the grill-determined spaces are precisely the subspaces of quotients of topological spaces

in P-Near (or S-Near or Grill). Next we show that the grill-determined spaces are precisely the quotients of subspaces of topological spaces in P-Near.

4.5. Definitions. A prenearness space is called subtopological iff it is a prenearness subspace of a topological space. The subcategory of P-Near whose objects are the subtopological spaces is denoted by SubTop.

4.6. Remarks. (1) The category SubTop was introduced by H.L. Bentley [3]. It is the intersection of the categories Grill and Near. SubTop is bireflective in Grill, and the SubTop-reflector is the restriction of the Near-reflector on S-Near.

(2) Top is bicoreflective in SubTop and closed under the formation of products in SubTop. Since Top is not cartesian closed, this implies SubTop is not cartesian closed (see L.D. Nel [20]).

4.7. Theorem. The grill-determined spaces are precisely the prenearness quotients of the subtopological spaces. Thus Grill is the coreflective hull in P-Near (or S-Near or Grill) of SubTop.

Proof. Similar to the proof of 4.1.

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