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A NOTE ON GEOMETRIC CHARACTERIZATION OF FRÉCHET DIFFERENTIABILITY

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Abstract: This note gives a direct geometric characterization of Fréchet differentiability of mappings between Banach spaces.

Key words: Banach space, Fréchet differentiability, cone.

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1. In the paper [1], the geometric characterization of Fréchet differentiability in Banach spaces by means of the notion of a tangent was given. The notion of a tangent in a Banach space was introduced there as a generalization of the Roetman's definition of a tangent in a finitely dimensional space (see [2]), i.e. using an intersection of a certain system of co-cones (see below).

Giving that geometric characterization of differentiability, we can avoid the notions of a tangent and a co-cone and deal with the intersection mentioned above as with an only basic notion. Of course, the principal ideas of that procedure remain the same as earlier, however, the characterization will be obtained from new, say, the pure geometric approximation point of view. This is the aim of our paper.

2. For completeness, let us recall first the notions of a co-cone and a circular co-cone that were introduced in the paper [1]:

Definition 1. Let X be a normed linear space, C a cone in X with a vertex at 0, H a linear subspace of X of the co-dimension 1 and denote $S_1 = \{x \in X : \|x\| = 1\}$. The number $\alpha = \operatorname{dist}(C \cap S_1, H)$ is called the deviation of C from H, the set $C' = X \setminus [C \cup (-C)]$ is called the co-cone to C (in X) and denoting by $\mathcal{C}_{H,\alpha}$ the system of all cones in X with a vertex at 0 and a deviation α from H, the set

$$C'_{H,\alpha} = \bigcap \{\overline{C'} : C' \text{ is a co-cone to some } C \in \mathcal{C}_{H,\alpha}\}$$

(it is a co-cone in X, too) is called the circular co-cone in X with the vertex 0 and the co-deviation α from H.

It is easy to see that

$$(1) \qquad C'_{H,\alpha} + \{\lambda x : \lambda \geq 0,\ x \in S_1,\ \operatorname{dist}(x,H) \leq \alpha\};$$

however, this is just the formula by which a circular co-cone was defined in [1] and hence, both the definitions of circular co-cones here and in [1] are equivalent.

Definition 2. Let X be a normed linear space, Z a linear subspace of X and $\varepsilon > 0$. The set

$$C(Z, \varepsilon) = \{x \in X: \text{dist}(x,Z) \leq \varepsilon \|x\|\}$$

is called the ε-cone of Z (in X).

Note that any ε-cone is a cone but it is not necessarily convex. There is a simple relation between ε-cones and cirvular co-cones.

<u>Lemma 1</u>. Let X be a Banach space, Z a closed linear subspace of X and $\varepsilon > 0$. Denote by $\mathcal{H}$ the system of all closed linear subspaces of X of the co-dimension 1. Then

$$C(Z, \varepsilon) = \bigcap \{C'_{H,\varepsilon} : Z \subset H \in \mathcal{H}\} .$$

<u>Proof</u>. Let $x \in C(Z, \varepsilon)$ be arbitrary and take an arbitrary $H \in \mathcal{H}$ containing Z. Then $\text{dist}(x,Z) \leq \varepsilon \|x\|$ and hence

$$\text{dist}\left(\frac{x}{\|x\|}, H\right) \leq \text{dist}\left(\frac{x}{\|x\|}, Z\right) \leq \varepsilon ,$$

i.e. $x \in C'_{H,\varepsilon}$ by (1). So we have proved that

$$C(Z, \varepsilon) \subset \bigcap \{C'_{H,\varepsilon} : Z \subset H \in \mathcal{H}\} .$$

On the other hand, let x_0 be an arbitrary point of $\bigcap \{C'_{H,\varepsilon} : Z \subset H \in \mathcal{H}\}$. We shall proceed similarly as in the respective part of the proof of Theorem 1 in [1]: Denoting by X^* the dual space of X, we set

$$Z^+ = \{x^* \in X^* : \|x^*\| = 1, \langle z,x^* \rangle = 0 \text{ whenever}$$

$z \in Z \}$

and $\mathfrak{X} = \{ H_{x^*} : x^* \in Z^+ \}$ where $H_{x^*} = \{ x \in X : \langle x, x^* \rangle = 0 \}$; it is $\mathfrak{X} = \{ H : Z \subset H \in \mathcal{H} \}$ and $\bigcap \{ H : H \in \mathfrak{X} \} = Z$. By [1], Lemma 1, we have

$$(2) \qquad |\langle x_0, x^* \rangle| \leq \frac{\varepsilon \cdot \langle u_{x^*}, x^* \rangle}{\text{dist}\,(u_{x^*}, H_{x^*})} \cdot \| x_0 \|$$

for all $x^* \in Z^+$ and $u_{x^*} \in X \setminus H_{x^*}$. According to Hahn-Banach Theorem, there exist $x_0^* \in X^*$ such that $\| x_0^* \| = 1$, $\langle z, x_0^* \rangle = 0$ whenever $z \in Z$ and

$$\langle x_0, x_0^* \rangle = \text{dist}\,(x_0, Z) .$$

It is $x_0^* \in Z^+$ and hence, for every given $\delta > 0$, choosing $u_{x_0^*} \in X \setminus H_{x_0^*}$ so that $\| u_{x_0^*} \| = 1$ and

$$\text{dist}\,(u_{x_0^*}, H_{x_0^*}) \geq \frac{\varepsilon}{\varepsilon + \delta}$$

(such $u_{x_0^*}$ exists by the well-known theorem of F. Riesz, see e.g. [3]), we obtain from (2)

$$|\langle x_0, x_0^* \rangle| \leq (\varepsilon + \delta) \| x_0 \| .$$

We conclude that

$$\text{dist}\,(x_0, Z) = \langle x_0, x_0^* \rangle \leq (\varepsilon + \delta) \| x_0 \| \text{ for all } \delta > 0 ,$$

i.e. $x_0 \in C(Z, \varepsilon)$ by Definition 2. The proof is completed.

<u>Definition 3</u>. Let X , Y be normed linear spaces, $L: X \longrightarrow Y$ a linear mapping and $\varepsilon > 0$. The set

$$C(L, \varepsilon) = \{ (x,y) \in X \times Y : \| y - Lx \| \leq \varepsilon \| (x,y) \| \}$$

is called the ε-cone of L (the norm on $X \times Y$ is given by $\| (x,y) \| = \| x \| + \| y \|$ (or by any other equivalent one)).

So, if L is a linear mapping from X into Y , we can consider two associated ε-cones: $C(L, \varepsilon)$ according to Definition 3 and $C(G(L), \varepsilon)$ according to Definition 2, where $G(L)$ denotes the graph of L in $X \times Y$. Both these ε-cones are in a close relation, as the following two lemmas show.

<u>Lemma 2</u>. $C(L, \varepsilon) \subset C(G(L), \varepsilon)$ for each $\varepsilon > 0$.

<u>Proof</u>. Let (x,y) be a point of $C(L, \varepsilon)$. Then

$$\text{dist}\,((x,y), G(L)) \leq \| (x,y) - (x,Lx) \| = \\ = \| y - Lx \| \leq \varepsilon \| (x,y) \| ,$$

i.e. $(x,y) \in C(G(L), \varepsilon)$.

<u>Lemma 3</u>. $C(L, \varepsilon) \supset C(G(L), \varepsilon')$ for each $\varepsilon, \varepsilon' > 0$ whenever $\varepsilon'(1 + \| L \|) < \varepsilon$.

<u>Proof.</u> Let (x,y) be in $C(G(L), \varepsilon')$ and let $\delta > 0$ be such that $(\varepsilon' + \delta)(1 + \| L \|) \leq \varepsilon$. Take $u \in X$ so that

$$\| (x,y) - (u,Lu) \| = \| (x - u, y - Lu) \| \leq \\ \leq (\varepsilon' + \delta) \| (x,y) \| .$$

Then

$$\| y - Lx \| \leq \| y - Lu \| + \| L(x - u) \| \leq$$

$$\leq \| (x - u, y - Lu) \| + \| L \| \cdot \| x - u \| \leq$$

$$\leq (\varepsilon' + \delta') \| (x,y) \| + \| L \| \cdot \| (x - u, y - Lu) \| \leq$$

$$\leq (\varepsilon' + \delta' + \| L \| (\varepsilon' + \delta')) \cdot \| (x,y) \| =$$

$$= (\varepsilon' + \delta')(1 + \| L \|) \| (x,y) \| \leq \varepsilon \| (x,y) \| ,$$

i.e. the point (x,y) is in $C(L, \varepsilon)$.

3. We are going now to our main theorems. These theorems can be derived (in Banach spaces) from [1], Theorem 1 and our Lemma 1; however, we prefer to present here the direct proofs of them.

Theorem 1. Let X , Y be normed linear spaces, $F: X \to Y$ a mapping Fréchet differentiable at 0 , $F(0) = 0$. Denote by $L = F'(0)$ the Fréchet derivative of F at 0 , by $R = F - L$ the remainder and set

$$\delta(\varepsilon) = \sup \{ \delta > 0 : \| Rx \| \leq \varepsilon \| x \| \text{ whenever } \| x \| \leq \delta , x \in X \}$$

for $\varepsilon > 0$. Then for each $\varepsilon > 0$,

$$G(F) \cap B_{X \times Y} (0, \delta(\varepsilon)) \subset C(G(L), \varepsilon)$$

where $B_{X \times Y}(0,r) = \{ z \in X \times Y : \| z \| \leq r \}$.

Proof. Let $\varepsilon > 0$ be arbitrary, let $(0,0) \neq (x, Fx) \in$

$\in G(F) \cap B_{X \times Y}(0, \delta(\varepsilon))$. Then $\|(x,Fx)\| \leq \delta(\varepsilon)$ and $\|x\| \leq \|(x,Fx)\| \leq \delta(\varepsilon)$, so that $\|Rx\| \leq \varepsilon \|x\|$. Set $\lambda = \|(x,Fx)\|$ and $(u,v) = \lambda^{-1}(x,Fx)$. Then $\|(u,v)\| = 1$ and

$$\begin{aligned} \operatorname{dist}((u,v), G(L)) &\leq \|(u,v) - (u,Lu)\| = \\ &= \|(0, v - Lu)\| = \|v - Lu\| = \\ &= \lambda^{-1} \|Fx - Lx\| = \lambda^{-1} \|Rx\| \leq \\ &\leq \lambda^{-1} \varepsilon \|x\| \leq \varepsilon . \end{aligned}$$

As $(x,Fx) = \lambda(u,v)$, we have

$$\operatorname{dist}((x,Fx), G(L)) \leq \lambda \varepsilon = \varepsilon \|(x,Fx)\| ,$$

i.e. $(x,Fx) \in C(G(L), \varepsilon)$.

Theorem 2. Let X , Y be normed linear spaces, $F: X \longrightarrow Y$ a mapping continuous at 0 with $F(0) = 0$. Let $L: X \longrightarrow Y$ be a continuous linear mapping such that for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$G(F) \cap B_{X \times Y}(0, \delta) \subset C(G(L), \varepsilon) .$$

Then F is Fréchet differentiable at 0 and $L = F'(0)$.

Proof. Let $\varepsilon' > 0$ be given and take $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{2(1 + \|L\|)} \quad \text{and} \quad \frac{2\varepsilon(1 + \|L\|)^2}{1 - 2\varepsilon(1 + \|L\|)} < \varepsilon' .$$

Take $\delta > 0$ so that

$$G(F) \cap B_{X \times Y}(0, \delta) \subset C(G(L), \varepsilon) .$$

As the mapping F is continuous at 0, there exists $\delta' \in (0, \delta)$ such that $x \in X$, $\| x \| \leq \delta'$ implies $\| (x, Fx) \| \leq \delta$.

Let $x \in X$ with $\| x \| \leq \delta'$ be given. Then $\| (x, Fx) \| \leq \delta$, hence

$$\operatorname{dist} \left(\frac{(x, Fx)}{\| (x, Fx) \|}, G(L) \right) \leq \varepsilon ,$$

and so we can find $x_\varepsilon \in X$ such that

$$\left\| \frac{(x, Fx)}{\| (x, Fx) \|} - (x_\varepsilon, Lx_\varepsilon) \right\| < 2 \varepsilon .$$

It follows now that

$$\| x - x_\varepsilon \| \leq 2 \varepsilon \| (x, Fx) \|$$

and

$$\| L(x - x_\varepsilon) + Rx \| = \| Fx - Lx_\varepsilon \| \leq 2 \varepsilon \| (x, Fx) \|$$

where $R = F - L$. Then

$$\begin{aligned} \| Rx \| &\leq \| L(x - x_\varepsilon) \| + 2 \varepsilon \| (x, Fx) \| \leq \\ &\leq \| L \| \cdot \| x - x_\varepsilon \| + 2 \varepsilon \| (x, Fx) \| \leq \\ &\leq 2 \varepsilon (1 + \| L \|) \| (x, Fx) \| = 2 \varepsilon (1 + \| L \|)(\| x \| + \\ &+ \| Fx \|) \leq 2 \varepsilon (1 + \| L \|)(\| x \| + \| L \| \| x \| + \\ &+ \| Rx \|) \end{aligned}$$

and hence,

$$\|Rx\| \le \frac{2\varepsilon(1+\|L\|)^2}{1-2\varepsilon(1+\|L\|)} \cdot \|x\| \le \varepsilon'\|x\| .$$

We have proved that for each $\varepsilon' > 0$, there is $\delta' > 0$ such that $x \in X$, $\|x\| \le \delta'$ implies $\|Rx\| = \|Fx - Lx\| \le \varepsilon'\|x\|$, which means that L is the Fréchet derivative of F at 0 .

4. At the end, the second author wishes to use his opportunity to make the following corrections of his paper [1]:

In the definition of a tangent on p. 526, the condition

" (iii) The mapping F is continuous at x_0 "

would be added; a similar correction is needed in the definition on p. 532. The proof of the formula (10) on p. 532 (starting from the choice of a δ on p. 531) is incorrect; however, that formula follows easily from the continuity of F at x_0 . Some other misprints occured in [1] are not essential and they can be corrected by the reader without any difficulties.

References

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