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NOTE ON HOMOMORPHISM INTERPOLATION THEOREM

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Abstract: A homomorphism interpolation theorem for societies and cohomomorphisms is proved. This extends similar theorems for graphs and complete partitions.

Key words: Society, partition, homomorphism.

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This note contains a method by means of which one can prove homomorphism interpolation theorems (of the type discussed in [1 - 3]). Particularly, we give a short proof of [2]. Our method is directly applicable to infinite objects, too. Moreover, we prove that with two exceptions there are always at least two different complete partitions. Let us remark that the proof is a suitable category-theory modification of [1].

Let $\mathcal{S}$ be the category whose objects are sets with a hereditary family of subsets $(S = \langle X, \mathfrak{M} \rangle$, $N \subseteq M \in \mathfrak{M} \Longrightarrow N \in \mathfrak{M})$ and morphisms are cohomomorphisms ($f: \langle X, \mathfrak{M} \rangle \longrightarrow \langle Y, \mathfrak{N} \rangle$ is a morphism iff $N \in \mathfrak{N} \Longrightarrow f^{-1}(N) \in \mathfrak{M}$). An object of category $\mathcal{S}$ is called a society, the members M of family $\mathfrak{M}$ are called teams. All the following considerations are done in the category $\mathcal{S}$.

We say that the society $R = \langle Y, \mathfrak{N} \rangle$ is inductive created by a morphism $f: S = \langle Y, \mathfrak{M} \rangle \longrightarrow R$, iff $N \subseteq Y$, $N \in \mathfrak{N} \Longleftrightarrow f^{-1}(N) \in \mathfrak{M}$. The morphism f is called an inductive morphism (for R).

Let S be a society, φS be a cardinality of a smallest set Y (as to the cardinality) for which there exists a couple $(f, \mathfrak{N})$ such that the society $R = \langle Y, \mathfrak{N} \rangle$ is inductively created by the morphism $f: S \longrightarrow R$. It is easy to prove that the composition of two inductive morphisms is an inductive morphism.

Let x, y be two different elements of a team of society $S = \langle X, \mathfrak{M} \rangle$. Let $S/x{\sim}y$ be the society inductively created by a canonical morphism $p_{xy}: S \longrightarrow X/x{\sim}y$ (where $x{\sim}y$ is the equivalence which identifies only two points: x, y).

<u>Lemma</u>: Let S be a society. Then $\varphi S \leq \varphi S/x{\sim}y \leq \leq \varphi S + 1$.

Proof: The inequality $\varphi S \leq \varphi S/x{\sim}y$ is trivial. We say that a society $S = \langle X, \mathfrak{M} \rangle$ is discrete iff $\mathfrak{M} = \{\{x\} \mid x \in X\}$. We sign D_n discrete society such that $\operatorname{card} D_n = n$. Observe that if $R = \langle Y, \mathfrak{N} \rangle$ is an inductively created society by a morphism $f: S \longrightarrow R$, $\operatorname{card} R = \varphi S$, then R is a discrete society (see the definition $S/x{\sim}y$).

Now we can prove the second inequality. We construct a morphism g from the society $S/x{\sim}y$ onto the discrete society T, $\operatorname{card} T \leq \varphi S + 1$: Let $f: S = \langle X, \mathfrak{M} \rangle \longrightarrow D_n = \langle [1,n], \{\{i\} \mid i \in [1,n]\}\rangle$ be a morphism. Define the mapping g by $g|_{X \setminus \{x,y\}} = f$, $g(x\sim y) = n + 1$. From this follows $\varphi S/x\sim y \leqq \varphi S + 1$.

<u>Theorem 1:</u> Let S be a society, $\varphi S = n$,let $m \geqq n$ be natural numbers and let $f: S \longrightarrow D_m$ be an inductive morphism. Then for each $n \leqq k \leqq m$ there exists an inductive morphism $h: S \longrightarrow D_k$.

Proof: Let $S = \langle X, \mathfrak{M} \rangle$, card $X < \omega_0$.
We decompose the given morphism f in the finite number of mappings $f_i: T_i \longrightarrow T_{i+1}$, $i \in [1,p]$, such that f_i is inductive morphism, $T_1 = S$, card T_i = card $T_{i+1}+1$ (every mapping f_i is of type p_{xy}).
Applying Lemma it must exist the company T_i in this decomposition, for which is $\varphi T_i = k$. The existence of inductive morphisms $h_1: S \longrightarrow T_i$, $h_2: T_i \longrightarrow D_k$ is evident. We put $h = h_1 h_2$.

$$\begin{tikzcd} S \arrow[r] \arrow[drrrr, "h"'] & S/x\sim y \arrow[r] & \ldots \arrow[r] & \ldots \arrow[r] & T_i \arrow[r] \arrow[d, "h_2"] & \ldots \arrow[r] & D_m \\ & & & & D_k & & \end{tikzcd}$$

Let card $X \geqq \omega_0$, let $g: S \longrightarrow D_n$, $f: S \longrightarrow D_m$ be inductive morphisms. Let $\{G_i\}_{i=1}$ and $\{F_j\}_{j=1}$ be two partitions of X corresponding to the kernels Ker g , Ker f . For every i, j with $G_i \cap F_j \neq \emptyset$ we choose a point $y_{ij} \in G_i \cap F_j$. We sign the set of all these points Y . Let the mapping $e: X \longrightarrow Y$ map each set $G_i \cap F_j$ onto the point y_{ij} . We sign R the inductively created society on the set Y by the mapping e . Clearly there exist inductive morphisms

g' (f' respectively) onto D_n (D_m respectively). They are defined by $f'e = f$, $g'e = g$.

$$\begin{tikzcd}
S \arrow[rr, "e"] \arrow[d, "g"'] \arrow[drr, "f" near end] & & R \arrow[d, "f'"] \arrow[dll, "g'"' near start] \arrow[dl, dashed, "h'"'] \\
D_n & D_k & D_m
\end{tikzcd}$$

Clearly card $R < \omega_0$, so by the first part of the proof there exists an inductive morphism $h': R \longrightarrow D_k$. We put $h = h'e$.

<u>Corollary:</u> For the given natural number k , $n < k < m$, there are at least two different inductive morphisms $h: S \longrightarrow D_k$, $h_1: S \longrightarrow D_k$.

Proof: We preserve the notation of the proof of Theorem 1. Let us choose a couple $\langle x,y \rangle \in$ Kerf - Kerh . Applying Lemma and Theorem for $S/x \sim y$ we obtain the inductive morphism $h_1': S/x \sim y \longrightarrow D_k$. We put $h_1 = h_1' p_{xy}$. Clearly $h \neq h_1$ (h is the morphism defined in the preceding proof).

<u>Definition:</u> We call the partition a complete $\mathfrak{M}$-partition of a set X iff it corresponds to the kernel of some inductive morphism from the society $\langle X, \mathfrak{M} \rangle$ in a discrete society.

Obviously this coincides with the definition of the complete $\mathfrak{M}$-partition given in [3]: a complete $\mathfrak{M}$-partition of X of order k is a partition $\{S_1, \dots, S_k\}$ of X such that each $S_i \in \mathfrak{M}$ and $S_i \cup S_j \notin \mathfrak{M}$ for $i \neq j$.

<u>Theorem 2:</u> Let $m > n$ be natural numbers, let $S =$

$= \langle X, \mathfrak{M} \rangle$ be a society and let there exist a complete $\mathfrak{M}$-partition of X into m (into n, respectively) classes. Then for each k, $n < k < m$, there exist at least two different complete $\mathfrak{M}$-partitions into k classes.

The proof follows immediately from Theorem 1 and its Corollary.

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