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REMARKS ON THE SOLVABILITY AND NONSOLVABILITY OF WEAKLY NONLINEAR EQUATIONS [1)]

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Abstract: The existence of the solution of the nonlinear operator equation

$$Ax + \alpha Sx^+ - \beta Sx^- + Gx = f$$

(where A is a Fredholm linear selfadjoint noninvertible operator in a real semiordered Hilbert space X , S is a linear completely continuous operator in X , G is a nonlinear mapping in X , for $x \in X$ it is $x^+ = \max(x,0)$, $x = x^+ - x^-$) is studied.

Key words: Nonlinear operator equation, solvability, nonsolvability, multiplicity of solutions.

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1. Assumptions, Notations. Until further comment, X, Z will denote real Hilbert spaces with norms $\|x\|_X$, $\|z\|_Z$, respectively. The inner product in X is denoted by (x_1,x_2) .

A subset C of Z is called a cone if it is closed, convex, invariant under multiplication by nonnegative numbers, and if $C \cap (-C) = \{0\}$.

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1) The results contained in this note were first presented by the author at the Summer School on "Nonlinear Analysis and Mechanics",September 1974, Stará Lesná near Poprad,Slovakia, and they are announced without the proofs in the third part of the paper [7].

Let C be a given cone in Z with the following properties:

(Z 1) If $z \in Z$ then there exists a uniquely determined couple $z^+, z^- \in C$ such that $z = z^+ - z^-$.

(Z 2) The mappings $z \longmapsto z^+$., $z \longmapsto z^-$ are lipschitzian, i.e., there exists $\wp > 0$ such that

$$\| z_1^+ - z_2^+ \|_Z \leq \wp \| z_1 - z_2 \|_Z \,, \quad \| z_1^- - z_2^- \|_Z \leq \wp \| z_1 - z_2 \|_Z$$

for each $z_1, z_2 \in Z$.

(Z 3) $X \subset Z$ and the identity mapping from X into Z is continuous. Denote by η its norm.

(A 1) Let A be a linear bounded selfadjoint operator from X into X with a closed range $R(A)$ and finite-dimensional null-space $N(A)$, $\dim N(A) \geqq 1$.

Let P be the orthogonal projection from X onto $N(A)$ and let $Q = I - P$ (I is the identity operator in X) , i.e., Q is the orthogonal projection from X onto $R(A)$. Under our assumptions there exists a linear continuous map (the so-called right inverse) $M: R(A) \longrightarrow R(A)$ satisfying

$$MAx = Qx \quad (x \in X) \,,$$
$$AMy = y \quad (y \in R(A)) \,.$$

Denote by $\| M \|$ the norm of M .

Let S be a linear continuous mapping from Z into C with the norm $\| S \|$ and suppose:

(S 1) The mappings $x \longmapsto Sx^+$, $x \longmapsto Sx^-$ are completely continuous operators from X into X .

2. First result

Lemma 1. Let α, β be real numbers and suppose that

(S 2) $$\inf_{\substack{h \in N(A) \\ \|h\|_X = 1}} (\alpha Sh^+ - \beta Sh^-, h) > 0 .$$

Then $\delta_0(\alpha, \beta) > 0$, where

$$\delta_0(\alpha, \beta) = \{\sup \| \delta \geqq 0; \inf_{\substack{v \in R(A) \\ \|v\|_X \leqq \delta}} \inf_{\substack{h \in N(A) \\ \|h\|_X = 1}} (\alpha S(h+v)^+ - \beta S(h+v)^-, h) > 0\} .$$

Proof. Suppose that there exists $\delta_n > 0$, $\lim_{n \to \infty} \delta_n = 0$, and $v_n \in R(A)$, $\|v_n\|_X \leqq \delta_n$, $h_n \in N(A)$, $\|h_n\|_X = 1$, such that

$$\lim_{n \to \infty} (\alpha S(h_n + v_n)^+ - \beta S(h_n + v_n)^-, h_n) = 0 .$$

Since $N(A)$ is finite-dimensional, we can suppose that the sequence h_n converges in the norm topology of X to $h_0 \in N(A)$, $\|h_0\|_X = 1$. The continuity of S and the assumption (Z 2) imply

$$(\alpha Sh_0^+ - \beta Sh_0^-, h_0) = 0$$

which contradicts (S 2).

Theorem 1. Let α, β be real numbers. Suppose (Z 1) - (Z 3), (A 1), (S 1), (S 2). Moreover, suppose that

(G 1) $G: X \longrightarrow X$ is a completely continuous operator such that

(G 2) $$\sup_{x \in X} \|Gx\|_X < \infty .$$

Then the equation

(1) $\quad Ax + \alpha Sx^{+} - \beta Sx^{-} + Gx = f$

is solvable in X for each $f \in X$ provided

$$(|\alpha| + |\beta|)\,\|S\|_{\varphi} < \frac{\delta_0(\alpha,\beta)}{1+\delta_0(\alpha,\beta)}\,\|M\|^{-1} .$$

Proof. According to [4, Theorem 1] it suffices to verify the following condition:

For any $K>0$ we have $t_K>0$ such that

(2) $$(\alpha S(t(h+v)^{+}) - \beta S(t(h+v)^{-}),h) + (G(t(h+v)),h) \geqq K$$

for all $t \geqq t_K$, $h \in N(A)$, $\|h\|_X = 1$, $v \in R(A)$, $\|v\|_X \leqq \delta$, where $\delta < \delta_0(\alpha,\beta)$.

The assumptions (S 2),(G 2) and Lemma 1 immediately imply (2).

Remark 1. The assumption (G 2) may be (without changing the proof) replaced by the growth condition $\|Gx\|_X \leqq c_1 + c_2 \|x\|_X^{\sigma}$, where $\sigma \in \langle 0,1)$. If the constant c_2 is sufficiently small, then it may be $\sigma = 1$ and the same assertion as in Theorem 1 is valid.

Remark 2. Suppose, moreover,

(A 2) $\quad \dim N(A) = 1$

and $N(A)$ is a linear hull of $h_0 \in X$, $\|h_0\|_X = 1$.

If we suppose

(S 3) $\quad (Sh_0^{+},h_0) = -(Sh_0^{-},h_0) \neq 0$

then $\delta_0(\alpha,\beta) = \delta_0(1,0)$ and if $\alpha \neq -\beta$ then the condition (S 2) is fulfilled.

3. Second result

Lemma 2. Let α, β be real numbers and suppose (A 1), (A 2) and

$$\text{(S 4)} \quad \begin{cases} (\alpha Sh_0^+ - \beta Sh_0^- , h_0) > 0 \\ (\beta Sh_0^+ - \alpha Sh_0^- , h_0) < 0 . \end{cases}$$

Then $\delta_1(\alpha,\beta) > 0$, where

$$\delta_1(\alpha,\beta) = \sup \{ \delta \geqq 0 \, ; \inf_{\substack{v \in R(A) \\ \|v\|_X \leqq \delta}} (\alpha S(h_0 + v)^+ - \beta S(h_0 + v)^- , h_0) > 0 \text{ and}$$

$$\sup_{\substack{v \in R(A) \\ \|v\|_X \leqq \delta}} (\beta S(h_0 + v)^+ - \alpha S(h_0 + v)^- , h_0) < 0 \} .$$

(The proof is quite analogous of that in Lemma 1.)

Theorem 2. Let α, β be real numbers and suppose: (Z 1) - (Z 3), (A1), (A 2), (S 4). Let $G: X \longrightarrow X$ be a lipschitzian mapping, i.e. there exists $c > 0$ such that

$$\text{(G 3)} \qquad \| Gx_1 - Gx_2 \|_X \leqq c \, \| x_1 - x_2 \|_X$$

for each $x_1, x_2 \in X$. Suppose (G 2).

If

$$(3) \quad k = \|M\| \, (\varphi \|S\| \eta \, (|\alpha| + |\beta|) + c) < 1 ,$$

$$(4) \quad \frac{k}{1-k} < \delta_1(\alpha,\beta) ,$$

then there exists a lower semicontinuous function Γ : $: R(A) \longrightarrow (-\infty, \infty)$, bounded from below on bounded subsets of $R(A)$ and with the following properties:

a) The equation (1) has a solution for the right hand side $f \in X$ if and only if

$$(f,h_0) \geqq \Gamma(Qf) .$$

b) The equation (1) has at least two solutions for the right hand side $f \in X$ provided

$$(f,h_0) > \Gamma(Qf) .$$

Proof of Theorem 2

Step 1. For fixed $t \in (-\infty,\infty) = R_1$ and $f \in X$ define the mapping

$$F_{t,f} : R(A) \longrightarrow R(A)$$

by

$$F_{t,f}: v \longmapsto -MQ\,(\alpha S(th_0 + v)^+ - \beta S(th_0 + v)^- +$$
$$+ G(th_0 + v) - f) .$$

With respect to (3) the mapping $F_{t,f}$ is lipschitzian with the constant $k<1$ and thus according to the Banach's contraction mapping principle there exists a unique fixed point $v_{t,f} \in R(A)$ of $F_{t,f}$.

Step 2. For all $t_1, t_2 \in R_1$ and $f_1, f_2 \in X$ we obtain (by an easy calculation)

$$\| v_{t_1,f_1} - v_{t_2,f_2} \|_X \leqq \frac{k}{1-k} |t_1 - t_2| + \frac{\|M\|}{1-k} \|f_1 - f_2\|_X \ .$$

Step 3. Define

$$\Phi_f : t \longmapsto P(\alpha S(th_o + v_{t,f})^+ - \beta S(th_o + v_{t,f})^- + G(th_o + v_{t,f})).$$

The equation (1) with the right hand side $f \in X$ has a solution $x_o \in X$ if and only if there exists $t_o \in R_1$ such that

(5) $$\Phi_f(t_o) = Pf \ .$$

Step 4. Since $v_{t,f} = v_{t,Qf}$ and $N(A)$ is one-dimensional, the equation (5) is equivalent with

(6) $$\varphi_{Qf}(t) = (f, h_o) \ ,$$

where

(7) $$\varphi_{Qf} : t \longmapsto (\Phi_f(t), h_o) \ .$$

Step 5. For fixed $f \in X$ the function φ_{Qf} is continuous on R_1 and, moreover:

$$\lim_{|t| \to \infty} \varphi_{Qf}(t) = \infty$$

(it follows from the inequalities

$$\varphi_{Qf}(t) \geqq t \ (\alpha S(h_o + \frac{v_{t,f}}{t})^+ - \beta S(h_o + \frac{v_{t,f}}{t})^-, h_o) - \sup_{x \in X} \|Gx\|_X \quad \text{if} \quad t > 0 \ ,$$

$$\varphi_{Qf}(t) \geqq -t \ (\beta S(h_o + \frac{v_{t,f}}{t})^+ - \alpha S(h_o + \frac{v_{t,f}}{t})^-, h_o) -$$

$$- \sup_{x \in X} \|Gx\|_X \quad \text{if} \quad t < 0$$

and from (S 4),(4), Lemma 2 and Step 2).

Step 6. Define

$$\Gamma(Qf) = \min_{t \in R_1} \varphi_{Qf}(t) .$$

The assertions of Theorem 2 follow now immediately from Steps 5,3,4 .

Remark 3. If $(Sh_0^+, h_0)^2 \neq (Sh_0^-, h_0)^2$ then there exist α, $\beta \in R_1$ satisfying (S 4).

Remark 4. Note that the typical example of the equation (1) is the boundary value problem for nonlinear elliptic partial differential equation of the order $2m$ (see [7]). In this case we put $X = W_0^{m,2}(\Omega)$ (the well-known Sobolev space) and $Z = L_2(\Omega)$. In the case of second order partial differential equations it is possible to put $X = Z = W_0^{1,2}(\Omega)$. In both cases C may be the set of all almost everywhere nonnegative functions from Z .

Remark 5. The second order ordinary differential equations of the type (1) are investigated in [6]. The generalization of the results from [6] as well as the study of periodic problems and partial differential equations of second order is in [2, 3] .

Remark 6. The analogous result as in Theorem 2 is proved in [1, 5] for partial differential equations of the second order in the case that h_0 is a nonnegative function. In this note the condition " $h_0 \geqq 0$ " is replaced by (S 4) (and, of course, the abstract consideration is useful for higher order equations). A. Ambrosetti communicated me that E. Podolak (Princeton University) prepares the manuscript in which the condition " $h_0 \geqq 0$ " is replaced in the case of second order partial differential equations by

$$\int_{\Omega} (h_0^+(x))^2 dx \neq \int_{\Omega} (h_0^-(x))^2 dx \ .$$

But in all previous papers (also in the present note) for the method of proofs it is essential that $\dim N(A) = 1$. The observation of the analogous problems as in Theorem 2 but in the case of $\dim N(A) > 1$ is a terra incognita by the author's best knowledge.

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