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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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NONCONFORM FINITE ELEMENT PROCEDURE FOR SOLVING OF THE SIMPLY SUPPORTED PLATE PROBLEM

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Abstract: This paper contains hybrid variational formulation of the simply supported plate problem and one example of finite element method based on this variational formulation. There are not imposed too strong continuity requirements upon the "test" functions.

Key words: Finite elements.

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§ 1. Introduction. Let us solve the following variational problem: Find $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that

(1) $$a(u, \varphi) = (f, \varphi)$$

for all $\varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ where $f \in L_2(\Omega)$ is a given function, Ω is a Lipschitz domain on the plane and

$$a(u, \varphi) = \int_\Omega \Big\{ \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_1^2} + 2(1-\sigma) \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \varphi}{\partial x_2^2} + \sigma \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_2^2} + \sigma \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \varphi}{\partial x_1^2} \Big\} dx_1\, dx_2 \quad \text{for fixed } \sigma \in (0,1) .$$

Remark: The problem above is a simple model of a simply supported plate.

Theorem 1. There exists the unique solution u of the problem (1) for arbitrary given $f \in L_2(\Omega)$.

Proof: see [3].

Notations and definitions:

1) Notation of all functional spaces see [3].

2) Let G be a domain of the plain. Then

$$\|w\|_{0,G} \overset{df}{=} \|w\|_{L_2(G)}, \quad |w|_{k,G} \overset{df}{=} \Big(\sum_{|\alpha|=k} \|D^{\alpha} w\|^2_{0,G}\Big)^{\frac{1}{2}},$$

$$\|w\|_{k,G} \overset{df}{=} \big(\|w\|^2_{0,G} + |w|_{k,G})^2\big)^{\frac{1}{2}}$$

for sufficiently smooth function w and for an integer k.

Let G be the boundary of G. Then

$\|w\|_{0,\partial G} \overset{df}{=} \|w\|_{L_2(\partial G)}$.

3) Let $\Omega_h = \{\Omega_{ih}\}_{i=1}^{m(h)}$ be a division of Ω defined for each fixed $h \in (0,1)$. This means: $\bar{\Omega} = \bigcup_{i=1}^{m(h)} \bar{\Omega}_{ih}$, $\Omega_{ih} \cap \Omega_{jh} = \emptyset$ for all $i \neq j$, $h \in (0,1)$.

4) Let $W^{2,2}(\Omega_h) = \{u \in L_2(\Omega) \mid u \in W^{2,2}(\Omega_{ih}),\ i = 1,\dots,m(h)\}$, $\mathcal{W}^{-2,2}(\Omega_h) = \{F \in (W^{2,2}(\Omega_h))'$, where $(W^{2,2}(\Omega_h))'$ is the dual space to $W^{2,2}(\Omega_h)$; $F(\varphi) = 0$ for all $\varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\}$.

§ 2. Hybrid variational formulation.

Hybrid problem: Find $\{u_h, F_h\} \in W^{2,2}(\Omega_h) \times$

$\times\ \mathcal{W}^{-2,2}(\Omega_{\mathcal{h}})$ such that

(2) $\quad A_h(u_h, F_h\, ;\, \varphi, G) = (f, \varphi)$

for all $\{\varphi, G\} \in W^{2,2}(\Omega_{\mathcal{h}}) \times \mathcal{W}^{-2,2}(\Omega_{\mathcal{h}})$, where

$$A_h(u_h, F_h\, ;\, \varphi, G) = \sum_{i=1}^{m(h)} \int_{\Omega_{ih}} \left\{ \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_1^2} + \right.$$

$$+ 2(1-\sigma) \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \varphi}{\partial x_2^2} +$$

$$\left. + \sigma \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_2^2} + \sigma \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \varphi}{\partial x_1^2} \right\} dx_1\, dx_2 + F_h(\varphi) +$$

$$+ G(u_h) .$$

<u>Remark</u>: About origin of the hybrid variational formulation see e.g. [4].

If we define $u_h = u$, $F_h(.) = (f,.) - a_h(u,.)$, where u is the solution of the problem (1), then it is easy to verify that $\{u_h, F_h\}$ solves the problem (2). It may be proved the uniqueness of this solution, nevertheless it will not be necessary further.

<u>Theorem 2</u>. Let $u \in W^{3,2}(\Omega)$. Then $\| Mu \|_{0,\partial\Omega} = 0$ and

(3) $$F_h(\varphi) \overset{\mathrm{df}}{=} (f,\varphi) - a_h(u,\varphi) = \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} Mu \frac{\partial \varphi}{\partial \nu} d\sigma$$

for $\varphi \in W^{2,2}(\Omega_{\mathcal{h}}) \cap W_0^{1,2}(\Omega)$, where $\nu = (\nu_1, \nu_2)$ is the outside normal vector and $Mu = -\sigma \Delta u -$

$$- (1-\sigma) \left(\frac{\partial^2 u}{\partial x_1^2} \nu_1^2 + 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \nu_1 \nu_2 + \frac{\partial^2 u}{\partial x_2^2} \nu_2^2 \right).$$

Proof: By simple using of Green's formula.

It is very natural to introduce the approximate Galerkin method as usual. Let S_h^1 and S_h^2 be subspaces of $W^{2,2}(\Omega_h)$ and $\mathcal{W}^{-2,2}(\Omega_h)$ algebraically for each fixed $h \in (0,1)$. We define the following semi-norms:

a) $|w|_{2,h} = \left(\sum_{i=1}^{m(h)} |w|_{2,\Omega_{ih}}\right)^{\frac{1}{2}}$ and $\|w\|_{2,h} =$

$= [\sum_{i=1}^{m(h)} (|w|_{2,\Omega_{ih}} + h^{-4}\|w\|_{0,\Omega_{ih}}]^{\frac{1}{2}}$ for each $w \in W^{2,2}(\Omega_h)$,

b) $|||F|||_{-2,h} = \sup_{\varphi \in S_h^1} |F(\varphi)|\,|\varphi|_{2,h}^{-1}$ for each $F \in \mathcal{W}^{-2,2}(\Omega_h)$. Let us suppose that $\{|u|_{2,h} = 0,\ u \in S_h^1 \Longrightarrow u \equiv 0$ on $\Omega\}$, i.e. $|\cdot|_{2,h}$ is a norm on S_h^1.

Approximate problem: Find $\{u_h^*, F_h^*\} \in S_h^1 \times S_h^2$ such that

(4) $\qquad A_h(u_h^*, F_h^*; \varphi, G) = (f, \varphi)$

for each $\{\varphi, G\} \in S_h^1 \times S_h^2$.

Lemma 1. There exists a constant C such that

(5) $\qquad \sup_{\mathcal{M}_h} |A_h(\psi, F; \varphi, G)| \geq C(|\psi|_{2,h}^2 + |||F|||_{-2,h}^2)^{\frac{1}{2}}$

for each $\{\psi, F\} \in S_h^1 \times S_h^2$, where $\mathcal{M}_h = \{(\varphi, G) \in$ $\in S_h^1 \times S_h^2,\ |\varphi|_{2,h}^2 + |||F|||_{-2,h}^2 = 1\}$.

Proof: Let $\{\psi, F\} \in S_h^1 \times S_h^2$ be given. We shall consider the following auxiliary problem: Find $\chi_h \in S_h^1$ such that $a_h(\varphi, \chi_h) = F(\varphi)$ for each $\varphi \in S_h^1$. There exists unique $\chi_h \in S_h^1$ for arbitrary given $F \in S_h^2$ and moreover

$$(6) \qquad c_1 \, |||F|||_{-2,h} \le |\chi_h|_{2,h} \le c_2 \, |||F|||_{-2,h}$$

where the constants c_1, c_2 are independent of F, h.
If we substitute $\varphi = \psi + \chi_h$ and $G = -2F$, we obtain $A_h(\psi, F; \psi+\chi_h, -2F) = a_h(\psi,\psi) + a_h(\chi_h, \chi_h)$. In accordance with (6) it holds

$$(7) \quad |A_h(\psi, F; \psi+\chi_h, -2F)| \ge C(|\psi|^2_{2,h} + |||F|||^2_{-2,h})$$

where C is independent of ψ, F, H. On the other hand, from (6) it follows

$$(8) \quad |\psi + \chi_h|^2_{2,h} + |||-2F|||^2_{-2,h} \le C(|\psi|^2_{2,h} + |||F|||^2_{-2,h}),$$

where C is independent of ψ, F, h. The assertion (5) is a simple consequence of (7) and (8),

<u>Theorem 3</u>. Let $\{u, F_h\}$ and $\{u_h^*, F_h^*\}$ be the solution of the problem (2) and (4) respectively. Then

$$(9) \quad |u - u_h^*|_{2,h} + |||F_h - F_h^*|||_{-2,h} \le C(\|u - v\|_{2,h} +$$

$$+ \sup_{G \in S_h^2} |G(u-v)| \, |||G|||^{-1}_{-2,h} + |||F_h - F|||_{-2,h},$$

where C is independent of u, F_h, $v \in S_h^1$, $F \in S_h^2$, u_h^*, F_h^*.

<u>Proof</u>: Let an arbitrary $v \in S_h^1$ and $F \in S_h^2$ be given.

Then it holds: $A_h(u_h^* - v, F_h^* - F; \varphi, G) = A_h(u - v; F_h - F; \varphi, G)$ for each $\{\varphi, G\} \in S_h^1 \times S_h^2$. We may estimate $\sup_{\mathcal{M}_h} |A_h(u - v, F_h - F; \varphi, G)|$ by the right hand side of the inequality (9). Finally, by using (5) we reach the following estimate:

$$(10) \quad |u_h^* - v|_{2,h} + |||F_h^* - F|||_{-2,h} \leq C(\|u - v\|_{2,h} +$$

$$+ \sup_{G \in S_h^2} |G(u - v)| \, |||G|||_{-2,h} + |||F_h - F|||_{-2,h}) ,$$

where C is independent of $u, v, F_h, F, u_h^*, F_h^*, h$. The assertion (9) follows from (10) immediately.

Remark: The semi-norm $|||\cdot|||_{-2,h}$ need not be equivalent to the natural "sup" norm in the space $W^{-2,2}(\Omega_h)$. From this point of view, Lemma 1 warrants the uniqueness of u_h^* only. In the following paragraph we give one example of a concrete numerical method. The main problem arising by the application of the error estimates (9), will be the proof that $|||\cdot|||_{-2,h}$ is a norm on the space S_h^2.

§ 3. Nonconform finite element method. Let Ω be a polygon. Let the division $\Omega_h = \{\Omega_{ih}\}_{i=1}^{m(h)}$ be a triangulation of Ω, i.e. there exists an affine mapping F_{ih} of the fix "reference" triangle $\mathcal{T}$ onto Ω_{ih}, $i = 1, \ldots, m(h)$. Let us denote by d_{ih} and φ_{ih} the diameter of Ω_{ih} and the diameter of the circle inscribed in Ω_{ih} respectively. Let us suppose that $d_h \stackrel{df}{=} \max_{i=1,\ldots,m(h)} d_{ih} = O(h)$ and $\varphi_h \stackrel{df}{=} \min_{i=1,\ldots,m(h)} \varphi_{ih}$, $\varphi_h^{-1} = O(h^{-1})$ for $h \to 0$. About details of this techniques see [5].

Remark: We shall define the spaces S_h^1, S_h^2 in such a way that $S_h^1 \not\subset W^{2,2}(\Omega)$. Corresponding finite element procedure is (in this sense) nonconform with respect to the classical variational principle. About advantages of such a method see [2] and [6].

Let $S^1 = \{\psi \mid \psi$ is a polynom of the 4th degree on $\mathcal{T}$; ψ is a polynom of the 3rd degree on each side of the triangle $\mathcal{T}\}$. Let us denote by A_k and a_k $(k = 1, 2, 3)$ vertices and mid-side-points of the reference triangle $\mathcal{T}$.

Lemma 2. An element ψ of S^1 has the following degrees of freedom: $D^{\alpha}\psi(A_i)$ for $|\alpha| \leq 1$ and $\frac{\partial \psi}{\partial \hat{\nu}_i}(a_i)$, $i = 1,2,3$, where $\hat{\nu}_i$ is an arbitrary vector which is not perpendicular to the corresponding outside normal vector ν_0 of the reference triangle $\mathcal{T}$.

Proof is based on the fact that ψ is on each "side" direction polynom of the 3rd degree.

Let $S_h^1 = \{\varphi \mid \varphi \circ F_{ih} \in S^1$ for $i = 1,\dots,m(h)$, if A is any vertex and a is any mid-side-point of any triangle $\Omega_{ih} \in \Omega_h$, then $D^{\alpha}\varphi$ is continuous in A (for $|\alpha| \leq 1$) and $\frac{\partial \varphi}{\partial \nu}$ in a (where ν is a normal vector direction at a) respectively, with respect to Ω; $\varphi \equiv 0$ on $\partial\Omega\}$. If A is a vertex of polygon Ω then necessarily $D^{\alpha}\varphi(A) = 0$ for $|\alpha| \leq 1$. Using this fact, it may be easily verified that $|\cdot|_{2,h}$ is a norm on S_h^1.

Let $\tilde{S}^2 = \{\mu \mid \mu$ is constant on each side of the triangle $\mathcal{T}\}$. We define $\tilde{S}_h^2 = \{\mu \mid \mu$ is a function on $\bigcup_{i=1}^{m(h)} \partial\Omega_{ih}$ such that $\mu \circ F_{ih} \in \tilde{S}^2$ for $i = 1,\dots$ $\dots, m(h)$; $\mu|_{\partial\Omega} \equiv 0\}$ with a norm $[\cdot]_{-2,h} \overset{df}{=} h^{\frac{1}{2}} (\sum_{i=1}^{m(h)} \|\cdot\|^2_{0,\partial\Omega_{ih}})$. Finally, let $S_h^2 =$ $= \{G \mid G \in \mathcal{W}^{-2,2}(\Omega_h)$ such that there exists $\mu \in \tilde{S}_h^2$,

$$G(\varphi) = \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} \mu \frac{\partial\varphi}{\partial\nu} \, d\sigma \quad \text{for each} \quad \varphi \in W^{2,2}(\Omega_h) \; .$$

We shall say that $G \in S_h^2$ and $\mu \in \tilde{S}_h^2$ are associated iff

$$G(\varphi) = \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} \mu \frac{\partial\varphi}{\partial\nu} \, d\sigma \quad \text{for each} \quad \varphi \in W^{2,2}(\Omega_h) \; .$$

Lemma 3. For each $u \in W^{k,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $k = 3,4$ there exists $v_h \in S_h^1$ such that

(11) $$\| u - v_h \|_{2,h} \le C \| u \|_{k,\Omega} h^{k-2}$$

where C is independent of u, h. The function v_h may be found such that it interpolates u at all degrees of freedom (i.e. $D^\alpha v_h(A) = D^\alpha u(A)$, $|\alpha| \le 1$ and $\frac{\partial v_h}{\partial\nu}(a) = \frac{\partial u}{\partial\nu}(a)$ at each vertex A and mid-side-point a of any triangle $\Omega_{ih} \in \Omega_h$).

Proof: It is based on the well-known Bramble-Hilbert's lemma (see [1]) and on the fact that a piecewise polynom φ of the 3rd degree, which satisfies boundary conditions $\varphi \equiv 0$ on $\partial\Omega$, may be interpolated exactly on each $\Omega_{ih} \in \Omega_h$.

The following three lemmas will be used to prove that $|||\cdot|||_{-2,h}$ is a norm on S_h^2 .

Lemma 4. Let $\mu \in \tilde{S}_h^2$ be given. Let us define $\varphi_\mu \in$ $\in S^1$ such that $D^\alpha \varphi_\mu(A_i) = 0$, $|\alpha| \le 1$, $i = 1,2,3$ and $\frac{\partial \varphi_\mu}{\partial \nu_0}(a_i) = \mu(a_i)$, $i = 1,2,3$. Then it follows

(12) $$\int_{\partial T} \mu \frac{\partial \varphi_\mu}{\partial \nu_0} d\sigma \ge C \|\mu\|_{0,\partial T} |\varphi_\mu|_{2,T}$$

where C is independent of μ and

(13) $$\|\mu\|_{0,\partial T} \asymp |\varphi_\mu|_{2,T} .$$

Proof: If (12) were not true then there would exist μ , $\|\mu\|_{0,\partial T} = 1$ - see assumptions of this lemma. But it may be easily computed that $\int_{\partial T} \mu \frac{\partial \varphi_\mu}{\partial \nu_0} d\sigma = 0$ iff $\mu(a_i) = 0$ for all $i = 1,2,3$.
The assertion((13) is an evident consequence of Lemma 2.

Lemma 5. Let $G \in S_h^2$ and $\mu \in S_h^2$ be associated. Then

(14) $$C\,[\mu]_{-2,h} \le |||G|||_{-2,h}$$

where the constant C is independent of G, μ, h .

Proof: Let $\mu \in \tilde{S}_h^2$ be given. Let φ belong to S_h^1 such that $D^\alpha \varphi(A) = 0$, $|\alpha| \le 1$ for each vertex A of any triangle $\Omega_{ih} \in \Omega_h$ and $\left|\frac{\partial \varphi}{\partial \nu}(a)\right| = h^{-1}/\mu(a)/$ for each mid-side-point a of any triangle $\Omega_{ih} \in \Omega_h$.

We introduce the following notation:

$$\hat{\mu}_{ih} \overset{df}{=} \mu \circ F_{ih}, \ \hat{\varphi}_{ih} \overset{df}{=} \varphi \circ F_{ih}, \ \hat{\nu}_{ih} \overset{df}{=} h \cdot F_{ih}^{-1} \nu \ .$$

Then it holds

$$(15) \int_{\partial\Omega_{ih}} \mu \frac{\partial \varphi}{\partial \nu} d\sigma \geq C h \int_{\partial \mathcal{T}} \hat{\mu}_{ih} \left(\frac{\partial \varphi}{\partial \nu} \circ F_{ih} \right) d\sigma = C \int_{\partial \mathcal{T}} \hat{\mu}_{ih} \frac{\partial \hat{\varphi}_{ih}}{\partial \hat{\nu}_{ih}} d\sigma \ ,$$

where C is independent of μ, φ, h. It may be shown that $\frac{\partial \hat{\varphi}_{ih}}{\partial \hat{\nu}_{ih}}(a) = \hat{\mu}_{ih}(a)$ and $\frac{\partial \hat{\varphi}}{\partial \mathcal{T}}(a) = 0$ at each mid-side-point a of the triangle $\mathcal{T}$, where τ is a tangential-direction.

According to the assumptions concerning triangulation Ω_h, we can estimate $|(\hat{\nu}_{ih}, \nu_0)_{\mathbb{R}_2}| \cdot \|\hat{\nu}_{ih}\|^{-1}_{\mathbb{R}_2} \geq \varepsilon > 0$ where ε does not depend on i, h. By an easy computation we obtain

$$\left| \frac{\partial \hat{\varphi}_{ih}}{\partial \nu_0}(a) \right| = |(\hat{\nu}_{ih}, \nu_0)_{\mathbb{R}_2}| \cdot \|\hat{\nu}_{ih}\|^{-1}_{\mathbb{R}_2} \cdot \left| \frac{\partial \hat{\varphi}_{ih}}{\partial \hat{\nu}_{ih}}(a) \right| \geq \varepsilon \left| \frac{\partial \hat{\varphi}_{ih}}{\partial \hat{\nu}_{ih}}(a) \right| .$$

By using (12) and (15) we reach

$$(16) \int_{\partial\Omega_{ih}} \mu \frac{\partial \varphi}{\partial \nu} d\sigma \geq C \|\hat{\mu}_{ih}\|_{0,\partial\mathcal{T}} |\varphi_{\hat{\mu}_{ih}}|_{2,\mathcal{T}}$$

where the constant C is independent of h, μ, φ. The estimate (14) follows from (16) immediately.

<u>Lemma 6</u>. Let $G \in S_h^2$ and $\mu \in S_h^2$ be associated. Then there exists a constant C independent of G, μ, h such that

$$(17) \qquad |G(\varphi)| \leq C [\mu]_{-2,h} \|\varphi\|_{2,h}$$

for each $\varphi \in W^{2,2}(\Omega_h)$.

Proof: Since $G(\varphi) = \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} \mu \frac{\partial \varphi}{\partial \nu} \, d\sigma$, we obtain that

$$|G(\varphi)| \le \Big(\sum_{i=1}^{m(h)} \|\mu\|^2_{0,\partial\Omega_{ih}}\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{m(h)} \Big\|\frac{\partial\varphi}{\partial\nu}\Big\|^2_{0,\partial\Omega_{ih}}\Big)^{\frac{1}{2}} .$$

Let us estimate $\Big\|\frac{\partial\varphi}{\partial\nu}\Big\|_{0,\partial\Omega_{ih}}$: $\Big\|\frac{\partial\varphi}{\partial\nu}\Big\|_{0,\partial\Omega_{ih}} \le C\, h^{-\frac{1}{2}} \|\hat{\varphi}\|_{2,\tau} =$ $= C\, h^{-\frac{1}{2}} (|\hat{\varphi}|^2_{2,\tau} + \|\hat{\varphi}\|^2_{0,\tau})^{\frac{1}{2}}$ where $\hat{\varphi} \overset{\text{df.}}{=} \varphi \circ F_{ih}$ and C is independent of φ, i, h. If we use the inverse affine mapping, we obtain the estimate $(|\hat{\varphi}|^2_{2,\tau} + \|\hat{\varphi}\|_{0,\tau})^{\frac{1}{2}} \le$ $\le C(h^2 |\varphi|^2_{2,\Omega_{ih}} + h^{-2} \|\hat{\varphi}\|_{0,\Omega_{ih}})^{\frac{1}{2}} = C\, h(|\varphi|^2_{2,\Omega_{ih}} +$ $+ h^{-4} \|\varphi\|^2_{0,\Omega_{ih}})^{\frac{1}{2}}$.

Lemma 7. Let us suppose that the solution u of the problem (1) belongs to $W^{k,2}(\Omega)$ for $k = 3,4$. Then there exists $v_k \in S_h^1$ such that

(18) $$\sup_{G \in S_h^2} | G(u - v_h) | \cdot |||G|||^{-1}_{-2,h} \le C \, \|u\|_{k,\Omega} \, h^{k-2}$$

where C is independent of u, h. The function v_h may be found such that it interpolates u at all degrees of freedom.

Proof: See (11), (14), (17).

It remains to analyse the norm $||| F_h - F |||_{-2,h}$. If $u \in W^{3,2}(\Omega)$ then we obtain (see (3)) that $F_h(\varphi) -$ $- F(\varphi) = \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} (Mu - \mu) \frac{\partial\varphi}{\partial\nu} \, d\sigma$, where $\mu \in \tilde{S}_h^2$

is associated with $F \in S_h^2$. Let us define an operator $\mathcal{L}_{ih}$ by the following way for each $i = 1,\dots,m(h)$: We denote by A_1^i, A_2^i, A_3^i and by a_1^i, a_2^i, a_3^i the vertices and the mid-side-points of the triangle Ω_{ih}. If v is a function given on side A_1^i, A_2^i, then $\mathcal{L}_{ih} v$ is a Lagrange interpolation at the points A_1^i, a_3^i, A_2^i. On the other sides of Ω_{ih} there is defined the operator $\mathcal{L}_{ih}$ by the similar way (i.e. as a three-point Lagrange interpolation). It may be easily verified that

$$\sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} (Mu-\mu)\frac{\partial\varphi}{\partial\nu}\,d\sigma = \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} (Mu-\mu)\left(\frac{\partial\varphi}{\partial\nu} - \mathcal{L}_{ih}\frac{\partial\varphi}{\partial\nu}\right)d\sigma$$

for $\varphi \in S_h^1$. Therefore it holds

$$(19)\ \|F_h - F\|_{-2,h} = \sup_{\varphi\in S_h^1} |\varphi|_{2,h}^{-1} \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} (Mu-\mu)\left(\frac{\partial\varphi}{\partial\nu} - \mathcal{L}_{ih}\frac{\partial\varphi}{\partial\nu}\right)d\sigma \,.$$

<u>Lemma 8</u>. Let us suppose that $w \in W^{4,2}(\Omega_{ih})$ and $Mw = 0$ on $\partial\Omega \cap \partial\Omega_{ih}$. Then there exists a constant C independent of w, i, h such that

$$(20)\ \int_{\partial\Omega_{ih}} (Mu-\mu_h(w))\left(\frac{\partial\varphi}{\partial\nu} - \mathcal{L}_{ih}\frac{\partial\varphi}{\partial\nu}\right)d\sigma \le C h^2 |w|_{4,\Omega_{ih}} |\varphi|_{2,\Omega_{ih}}$$

for each $\varphi \in S_h^1$, where $\mu_h(w) \stackrel{df.}{=} \left(\frac{3}{2} - 2\alpha\right)$
$\lim_{t\to 1_-} Mw(t A_k^i + (1-t)A_\ell^i) + \left(-\frac{1}{2} + 2\alpha\right)\lim_{t\to 0_+} Mw(tA_k^i +$
$+ (1-t)A_\ell^i)$ on each side $A_k^i A_\ell^i$, $k > \ell$, of the triangle Ω_{ih} for either $\alpha = 0$ or $\alpha = 1$.

<u>Proof</u>: Using the well known technique of the affine

mapping T_{ih} - see [5], it may be shown that

$$(21) \qquad \| Mw - \mu_h(w) \|_{0,\partial\Omega_{ih}} \leq C h^{-\frac{3}{2}} \| \hat{w} \|_{4,\tau}$$

and

$$(22) \qquad \left\| \frac{\partial \varphi}{\partial \nu} - \mathcal{L}_{ih} \frac{\partial \varphi}{\partial \nu} \right\|_{0,\partial\Omega_{ih}} \leq C h^{-\frac{1}{2}} \| \hat{\varphi} \|_{2,\tau} \quad ,$$

where $\hat{w} \overset{df.}{=} w \circ F_{ih}$, $\hat{\varphi} \overset{df.}{=} \varphi \circ F_{ih}$ and C is a constant independent of w, φ, i, h .

Let $\hat{\psi}$ and $\hat{\chi}$ be a polynom of the 1st and 3rd degree respectively. Then $\psi \overset{df.}{=} \hat{\psi} \circ F^{-1}_{ih}$ and $\chi \overset{df.}{=} \hat{\chi} \circ F^{-1}_{ih}$ is a polynom of the 1st and 3rd degree respectively. First we shall prove that

$$(23) \quad \begin{cases} \int_{\partial\Omega_{ih}} (M(w+\chi) - \mu_h(w+\chi)) \left(\frac{\partial(\varphi+\psi)}{\partial\nu} - \mathcal{L}_{ih} \frac{\partial(\varphi+\psi)}{\partial\nu} \right) d\sigma = \\ \int_{\partial\Omega_{ih}} (Mw - \mu_h(w)) \left(\frac{\partial\varphi}{\partial\nu} - \mathcal{L}_{ih} \frac{\partial\varphi}{\partial\nu} \right) d\sigma \ . \end{cases}$$

Because $\frac{\partial\psi}{\partial\nu} - \mathcal{L}_{ih}\frac{\partial\psi}{\partial\nu} \equiv 0$ on $\partial\Omega_{ih}$, it is sufficeint to verify that $\int_{\partial\Omega_{ih}} (M\chi - \mu_h(\chi)) \left(\frac{\partial\varphi}{\partial\nu} - \mathcal{L}_{ih} \frac{\partial\varphi}{\partial\nu} \right) d\sigma = 0$.

Let a side $A^i_k A^i_\ell$ of Ω_{ih} be given. We define

$(M\chi = \mu_h(\chi)) \left(\frac{\partial\varphi}{\partial\nu} - \mathcal{L}_{ih} \frac{\partial\varphi}{\partial\nu} \right)$ equals to $K(t)$ at the point $x = t A^i_\ell + (1-t) A^i_k$ for $t \in \langle 0,1 \rangle$. The function $K(t)$ is a polynom of the 4th degree and the numbers 0, 0.5, 1, 1.5 are roots of $K(t)$. Hence $\int_0^1 K(t)dt = 0$ and the equality (23) holds.

From (21), (22), (23) it follows that the left hand side of (20) is bounded by $C\,h^{-2} \inf_{\hat{\chi}} \| \hat{w} + \hat{\chi} \|_{4,\mathcal{T}} \cdot \inf_{\hat{\psi}} \| \hat{\varphi} + \hat{\psi} \|_{2,\mathcal{T}}$. This term may be bounded by $C\,h^{-2} | \hat{w} |_{4,\mathcal{T}} \cdot | \hat{\varphi} |_{2,\mathcal{T}}$. Transforming $\mathcal{T}$ into Ω_{ih}, we obtain the estimate (12).

By the same technique may be proved the following

Lemma 9. Let us suppose that $w \in W^{3,2}(\Omega_{ih})$. Then

$$(24) \quad \int_{\partial\Omega_{ih}} \mathcal{M} w \left(\frac{\partial \varphi}{\partial \nu} - \mathcal{L}_{ih} \frac{\partial \varphi}{\partial \nu} \right) d\sigma \leq C h |w|_{3,\Omega_{ih}} |\varphi|_{2,\Omega_{ih}}$$

for each $\varphi \in S_h^1$.

As a simple consequence of (19), (20), (24) we may obtain the following assertion:

Lemma 10. Let the solution $u \in W^{k,2}(\Omega)$, $k = 3,4$. Then there exists $G_h \in S_h^2$ for each $h \in (0,1)$ such that

$$(25) \quad ||| F_h - G_h |||_{-2,h} \leq C \| u \|_{k,\Omega}\, h^{k-2}$$

where the constant C is independent of u, h. The functional G_h may be found by the following way: If $k = 3$ then $G_h \equiv 0$. If $k = 4$ then G_h is associated with such a $\mu_h \in \tilde{S}_h^2$ which equals to $\mu_h(u)$ on each Ω_{ih} - see Lemma 8.

Now it is easy to prove some assertions concerning error estimates of the presented method.

Theorem 4. Let the solution u of the problem (1) belong to $W^{k,2}(\Omega)$, $k = 3,4$. Then

$$| u - u_h^* |_{2,h} = O(h^{4-k}) .$$

__Proof__: See (9), (11), (18), (25) .

__Theorem 5__. Let the solution u of the problem (1) belong to $W^{4,2}(\Omega)$. Then

$$\sup_{a \in \mathcal{N}_h} | Mu(a) - \mu_h^*(a) | = O(h^{\frac{1}{2}})$$

where μ_h^* is associated with F_h^* , $Mu(a)$ and $\mu_h^*(a)$ is the value of Mu and μ_h^* at the point a , $\mathcal{N}_h =$ $= \{a;\ a$ is mid-side point of $\Omega_{ih} \in \Omega_h \}$.

__Proof__: Let μ_h be the element of $\tilde{S}_h^2$ defined in Lemma 10 (case k = 4). Using (10), (14), (11), (18), (25) we reach the following estimate: $[\mu_h^* - \mu_h]_{-2,h} = O(h^2)$, i.e.

$$\sum_{a \in \mathcal{N}_h} | \mu_h^*(a) - \mu_h(a) | = O(h^{\frac{1}{2}}) \tag{26}$$

where $\mu_h(a)$ is the value of μ_h at the point a . But μ_h may be defined by two ways (see Lemma 10 and 8). If we replace μ_h by its average (i.e.

$$\mu_h = \frac{1}{2} (\lim_{t \to 0_+} Mu(t\, A_k^i + (1 - t)A_\ell^i) +$$

$$+ \lim_{t \to 1_-} Mu\, (t\, A_k^i + (1 - t)A_\ell^i\,))$$

on each side $A_k^i A_\ell^i$ of the triangle Ω_{ih}), we obtain the same estimate (26) evidently. On the other hand, it may be shown that

$$| \frac{1}{2} (\lim_{t \to 0_+} Mu(t\, A_k^i + (1 - t)A^i\,) + \lim_{t \to 1_-} Mu(t\, A_k^i +$$

$$+ (1 - t)A_\ell^i\,)) - Mu(\frac{1}{2} (A_k^i + A_\ell^i\,))| \leq C\, h\, |u|_{4,\Omega_{ih}}$$

on each side $A_k^i A_\ell^i$ of Ω_{ih} where C is independent of u, i, h . This finished the proof.

Remark: Using so called "Nitsche's trick", we would obtain error estimates of $u - u_h^*$ in L_2-norm. Since we cannot expect "a priori" better smoothness of u then $u \in W^{2,2}(\Omega)$ for even analytic function f , the L_2 estimate may be found localy only. Local error analysis of this case exceeds the purpose of this paper and will be published later on.

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