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CONCERNING THE RATE OF CONVERGENCE OF NEWTON'S PROCESS

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Abstract: The author establishes an explicit formula for the partial sums $x + \omega(x) + \dots + \omega^{(n)}(x)$ where ω is the rate of convergence obtained in [5] for the Newton's process

$$\omega(x) = \frac{x^2}{2(x^2 + d)^{1/2}} .$$

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In a recent series of investigations the author has proposed a new way of estimating convergence of iterative processes. Instead of defining the rate of convergence as a number, the author introduces the following

(1.1) Definition. Let T be an interval of the form $T = \{t;\ 0 < t < t_0\}$ for some positive t_0. A rate of convergence on T is a function ω defined on T with the following properties

1° ω maps T into itself

2° for each $t \in T$ the series

$$t + \omega(t) + \omega^2(t) + \dots$$

is convergent.

We use the abbreviation ω^n for the n-th iterate of the function ω , so that $\omega^2(t) = \omega(\omega(t))$ and so on. The sum of the above series will be denoted by σ . The function σ satisfies the following equation

$$\sigma(t) - t = \sigma(\omega(t)) .$$

In [5] we have applied the method of nondiscrete mathematical induction to Newton's process and obtained the rate of convergence together with the corresponding sigma function. The rate of convergence yields sharp estimates for each step of the process. However, to estimate the distance from the solution it is necessary to have explicit expressions for partial sums of the series $x + \omega(x) + \dots$. For the rate of convergence described in the Gatlinburg Lecture [4] such a formula has been recently established [6].

In the case of Newton's process, the rate of convergence turns out to be

$$\omega(x) = \frac{x^2}{2(x^2 + d)^{1/2}}$$

where d is a nonnegative number determined by the characteristics of the process [5]. The case $d = 0$ presents no difficulties, the function being linear. We restrict our attention to the case $d > 0$. To compute directly the superpositions $\omega^{(n)}(x)$ seems to be difficult, the expressions become complicated; on the other hand using the method suggested in [6], it is possible to establish an explicit formula for the partial sums $x + \omega(x) + \dots + \omega^n(x)$.

We begin with two lemmas concerning a recursively defined sequence.

(1.1) Let $x_0 > 1$. Define a sequence x_n by the recursive relation

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{1}{x_n} \right) .$$

Then

$$x_n = \frac{(x_0 + 1)^{2^n} + (x_0 - 1)^{2^n}}{(x_0 + 1)^{2^n} - (x_0 - 1)^{2^n}} .$$

Proof. Clearly it is sufficient to verify this formula inductively. We intend, however, to describe a heuristic approach to the result. We look for solutions of the form $x_n = \frac{u_n}{v_n}$; the relation to be satisfied becomes

$$\frac{u_{n+1}}{v_{n+1}} = \frac{1}{2} \left(\frac{u_n}{v_n} + \frac{v_n}{u_n} \right) = \frac{1}{2} \frac{u_n^2 + v_n^2}{u_n v_n} =$$

$$= \frac{(u_n + v_n)^2 + (u_n - v_n)^2}{(u_n + v_n)^2 - (u_n - v_n)^2} .$$

Upon setting $u_n + v_n = p_n$, $u_n - v_n = q_n$, we may reformulate the relation in the following form

$$\frac{p_{n+1} + q_{n+1}}{p_{n+1} - q_{n+1}} = \frac{p_n^2 + q_n^2}{p_n^2 - q_n^2} .$$

This will be satisfied if we set $p_{n+1} = p_n^2$ and $q_{n+1} = q_n^2$. Hence

$$x_n = \frac{p^{2^n} + q^{2^n}}{p^{2^n} - q^{2^n}}$$

for suitable p and q. A possible choice is to take p and q such that $p + q = x_o$ and $p - q = 1$. This leads to the formula

$$x_n = \frac{(x_o + 1)^{2^n} + (x_o - 1)^{2^n}}{(x_o + 1)^{2^n} - (x_o - 1)^{2^n}} \quad .$$

(1.2) Suppose that $y_o > d^{1/2}$ and that the sequence y_n is defined by the recursive formula

$$y_{n+1} = \frac{1}{2} \left(y_n + \frac{d}{y_n}\right)$$

then

$$y_n = d^{1/2} \; \frac{(y_o + d^{1/2})^{2^n} + (y_o - d^{1/2})^{2^n}}{(y_o + d^{1/2})^{2^n} - (y_o - d^{1/2})^{2^n}} \quad .$$

<u>Proof</u>. If we set $y_n = d^{1/2}\, x_n$ then x_n satisfies the relation

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{1}{x_n}\right) ;$$

the result follows from the preceding lemma by an elementary argument.

In the second part we apply the methods of [5] and [6] to the case of Newton's process.

(2.1) <u>Theorem</u>. Let $d > 0$. Then $x \longrightarrow \dfrac{x^2}{2(x^2 + d)^{1/2}}$ is a rate of convergence on the whole positive axis. For each natural n and each $x > 0$ we have

$$x + \omega(x) + \ldots + \omega^{(n)}(x) =$$

$$= x + (x^2 + d)^{1/2} - d^{1/2} \; \frac{\left(d^{1/2} + (x^2 + d)^{1/2}\right)^{2^{n+1}} + x^{2^{n+1}}}{\left(d^{1/2} + (x^2 + d)^{1/2}\right)^{2^{n+1}} - x^{2^{n+1}}} \, .$$

<u>Proof</u>. Let f be the function defined, for real x , by the formula

$$f(x) = x^2 - d \, .$$

Consider a point $x_0 > d^{1/2}$ and the Newton process for f starting at x_0 . Since

$$\frac{f(x)}{f'(x)} = \frac{1}{2} \left(x - \frac{d}{x}\right)$$

the Newton's process transforms a point $z \neq 0$ into the point

$$N(z) = z - \frac{f(z)}{f'(z)} = \frac{1}{2} \left(z + \frac{d}{z}\right) .$$

Suppose now that x_0 is such that $x_0 - N(x_0) = x$. It follows from Lemma (2.1) of[5] that

$$x_0 - N^n(x_0) = x + \omega(x) + \ldots + \omega^{(n-1)}(x) \, .$$

The equation to be satisfied by x_0 is

$x = x_0 - N(x_0) = \dfrac{f(x_0)}{f'(x_0)} = \dfrac{1}{2} \left(x_0 - \dfrac{d}{x_0}\right)$. It follows that

$x_o = x + (x^2 + d)^{1/2}$ whence, using Lemma (1.2)

$$N^n(x_o) = d^{1/2} \frac{(x_o + d^{1/2})^{2^n} + (x_o - d^{1/2})^{2^n}}{(x_o + d^{1/2})^{2^n} - (x_o - d^{1/2})^{2^n}} =$$

$$= d^{1/2} \frac{\left(\dfrac{d^{1/2} + (x^2 + d)^{1/2}}{x}\right)^{2^n} + 1}{\left(\dfrac{d^{1/2} + (x^2 + d)^{1/2}}{x}\right)^{2^n} - 1} .$$

This proves the theorem.

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