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Chaitan P. Gupta

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NONLINEAR EQUATIONS OF URYSOHN'S TYPE IN A BANACH SPACE

Chaitan P. GUPTA, Dekalb

Abstract: Let X be a real Banach space, X^* its dual Banach space. Let $K_1,\dots,K_n$ be a given finite family of compact monotone linear mappings from X^* into X and $F_1,\dots,F_n$ be a corresponding family of bounded demicontinuous mappings from X into X^*. Some results on the existence of solutions of the equation $u + \sum_{i=1}^{m} K_i F_i u = 0$ in X are obtained in this paper using Leray-Schauder Principle.

Key words and phrases: Urysohn's equations, compact mappings, angle-bounded mappings, Leray-Schauder Principle.

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Let X be a real Banach space and X^* its dual Banach space. Let $\{K_1,\dots,K_n\}$ be a finite family of linear mappings from X^* into X and $\{F_1,\dots,F_n\}$ be a corresponding family of (nonlinear) mappings from X into X^*. In this paper we establish some results on the existence of solutions for the nonlinear equation

$$u + \sum_{i=1}^{m} K_i F_i u = 0 \tag{1}$$

in the Banach space X. When the linear mappings $K_1,\dots,K_n$ are angle-bounded, equation (1) was studied by Browder [4] in the non-compact case and recently by Joshi [8] in the compact case. We study equation (1) when $K_1,\dots,K_n$ are compact monotone linear mappings and our main tool is the Leray-Schauder Principle ([9]): If C is a compact continuous mapping from a Banach space X into itself and there exists an $R > 0$ such that $u + tCu \neq 0$ for every $t \in [0,1]$ and every $u \in X$ with $\|u\| = R$, then there exists at least one solution u of the equation $u + Cu = 0$ in X with $\|u\| < R$. We do not use splitting lemma for angle-bounded linear mappings due to Browder-Gupta [5] and existence theorems for mappings of monotone type ([3],[6]) as in [8].

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<u>Main results</u>. Let X be a real Banach space and X^* its dual Banach space. We denote by (w,u) the duality pairing between the elements w in X^* and u in X. A bounded linear mapping $K: X \to X^*$ is said to be monotone if $(Ku,u) \geq 0$ for all u in X. The bounded linear monotone mapping is said to be <u>angle-bounded</u> if there exists a constant $\alpha \geq 0$ such that $|(Ku,v) - (Kv,u)| \leq 2\alpha \sqrt{(Ku,u)}\sqrt{(Kv,v)}$ for all u,v in X. A mapping K is said to be <u>compact</u> if it maps bounded subsets of X into relatively compact subsets of X^*. A mapping $F: X \to X^*$ is said to be <u>demi-continuous</u> if it is continuous from X to X^* endowed with weak-topology and F is said to be <u>bounded</u> if it maps bounded subsets of X^* into bounded sets of X^*.

Theorem 1 : Let $\{K_1,\dots,K_n\}$ *be a finite family of compact monotone linear mappings from* X^* *into* X *and let* $\{F_1,\dots,F_n\}$ *be a corresponding finite family of demi-continuous bounded (nonlinear) mappings from* X *into* X^*. *Suppose that there exists an* $R > 0$ *such that for any* n-*tuple* $\{u_1,\dots,u_n\}$ *in* X *with* $\sum_{i=1}^{n} \|u_i\|_X^2 = R^2$ *we have*

$$\sum_{i=1}^{n} (F_i u, u_i) \geq 0 \tag{2}$$

where $u = \sum_{i=1}^{n} u_i$.

Then the equation $u + \sum_{i=1}^{n} K_i F_i u = 0$ *has at least one solution* u *in* X .

Proof. We first observe that there exists a bounded continuous mapping $S: X \longrightarrow X^*$ such that for all u in X we have $\|Su\|_{X^*} \leq \|u\|_X$ and $(Su,u) \geq \frac{1}{2} \|u\|_X^2$. The existence of such an S was first observed by Amann [2] using an argument on partitions of unity due to Stanley-Weiss. Let, now, $Y = \underbrace{X \times \dots \times X}_{n}$ be the cartesian product of X with itself n-times and let for $U = [u_1,\dots,u_n] \in Y$, $\|U\|_Y = \sqrt{\sum_{i=1} \|u_i\|_X^2}$.

For each $\varepsilon > 0$ we define a mapping $T_\varepsilon : Y \longrightarrow Y$ by $T_\varepsilon(U) = [K_1F_1u + \varepsilon K_1 S u_1, \dots, K_nF_nu + \varepsilon K_n S u_n]$ where $U = [u_1,\dots,u_n] \in Y$, $u = \sum_{i=1}^{n} u_i$. Obviously T_ε is a compact continuous mapping from Y into Y . We assert that there exists a $U_\varepsilon \in Y$, $\|U_\varepsilon\| < R$ such that $(I + T_\varepsilon)(U_\varepsilon) = 0$, where I denotes the identity mapping on Y . Indeed, our assertion would follow from the Leray-Schauder Principle if we showed that $(I + tT_\varepsilon)(U) \neq 0$ for $t \in [0,1]$ and $U \in Y$ with $\|U\|_Y = R$. Now, clearly $(I + tT_\varepsilon)(U) \neq 0$ for $t = 0$ and $U \in Y$ with $\|U\|_Y = R$. For $t > 0$, let us suppose on the other hand that there exists a $U \in Y$, $\|U\|_Y = R$ such that $(I + tT_\varepsilon)U = 0$, i.e. $[u_1 + tK_1F_1u + t\varepsilon K_1Su_1, \dots, u_n + tK_nF_nu + t\varepsilon K_nSu_n] = 0$ where $U = [u_1, \dots, u_n]$ and $u = \sum_{i=1}^{n} u_i$. We then have that

$$0 = \sum_{i=1}^{n} (F_iu + \varepsilon Su_i,\ u_i + tK_iF_iu + t\varepsilon K_iSu_i)$$

$$\geq \sum_{i=1}^{n} (\varepsilon Su_i, u_i) \geq \frac{\varepsilon}{2} \sum_{i=1}^{n} \|u_i\|_X^2 = \frac{\varepsilon}{2} R^2 > 0$$

which is a contradiction. Hence $(I + tT_\varepsilon)(U) \neq 0$ for every $t \in [0,1]$ and every $U \in Y$ with $\|U\|_Y = R$ and thus there exists a $U_\varepsilon \in Y$ with $\|U_\varepsilon\|_Y < R$ and $(I + T_\varepsilon)(U_\varepsilon) = 0$.

Let, now, $T: Y \to Y$ be defined by $T(U) = [K_1F_1u, \dots, K_nF_nu]$ where $U = [u_1, \dots, u_n] \in Y$ and $u = \sum_{i=1}^{n} u_i$. Clearly T is a compact continuous mapping from Y into Y. Now,

$$0 = (I + T_\varepsilon)(U_\varepsilon) = (I + T)U_\varepsilon + \varepsilon W_\varepsilon$$

where $W_\varepsilon = [K_1Su_1^\varepsilon, \dots, K_nSu_n^\varepsilon]$ where $U_\varepsilon = [u_1^\varepsilon, \dots, u_n^\varepsilon]$. Clearly, $\{W_\varepsilon\}$'s are bounded in Y and so $\varepsilon W_\varepsilon \to 0$ strongly in Y. Hence $(I + T)U_\varepsilon \to 0$ strongly in Y. Since $\{U_\varepsilon\}$'s are bounded in Y and T is compact we see that there exists a sequence $\{\varepsilon_m\}$, $\varepsilon_m \to 0$ and a

$W \in Y$ such that $TU_{\varepsilon_m} \longrightarrow W$ strongly in Y. We then have that $U_{\varepsilon_m} \longrightarrow -W$ strongly in Y which implies by the continuity of T that $TU_{\varepsilon_m} \longrightarrow T(-W)$ strongly in Y and again since $(I+T)U_{\varepsilon} \longrightarrow 0$ strongly in Y as $\varepsilon \longrightarrow 0$ we have $U_{\varepsilon_m} \longrightarrow -T(-W)$ strongly in Y. Thus we must have $-W = -T(-W)$. Taking $U = -W$ we then get that $U + TU = 0$, that is $[u_1 + K_1F_1u, \dots, u_n + K_nF_nu] = 0$ where $U = [u_1, \dots, u_n]$ and $u = \sum_{i=1}^{n} u_i$. This immediately implies that $u + \sum_{i=1}^{n} K_iF_iu = 0$. Hence the Theorem. Q.E.D.

<u>Remark 1</u>. In the case $n = 1$, Theorem 1 is essentially due to Amann [2] (see also [1],[4],[7]).

<u>Remark 2</u>. If in Theorem 1, above we replace the demicontinuity of the F_i's by continuity we need not assume that the monotone mappings K_i are linear so long as we assume that they are Lipschitzian and $K_i(0) = 0$ for each i.

<u>Theorem 2</u>. <u>Let</u> $\{K_1, \dots, K_n\}$ <u>be a finite family of compact linear mappings from</u> X^* <u>into</u> X <u>such that there exists a constant</u> $\alpha > 0$ <u>with</u> $(w, K_iw) \geq \alpha \| K_iw \|_X^2$ <u>for</u> w <u>in</u> X^* <u>and</u> $i = 1,2,\dots,n$. <u>Let</u> $\{F_1, \dots, F_n\}$ <u>be the corresponding family of demi-continuous bounded (nonlinear) mappings from</u> X <u>into</u> X^*. <u>Suppose that there exists a</u> $\beta > 0$ <u>with</u> $\beta < \alpha$ <u>such that for any</u> n<u>-tuple</u> $\{u_1, \dots, u_n\}$ <u>in</u> X <u>we have</u>

$$(3) \qquad \sum_{i=1}^{n} (F_iu, u_i) \geq -\beta \sum_{i=1}^{n} \| u_i \|_X^2 + (F_i(0), u_i)$$

where $u = \sum_{i=1}^{n} u_i$.

Then the equation $u + \sum_{i=1}^{n} K_i F_i u = 0$ *has at least one solution* u *in* X .

Proof. Let $Y = \underbrace{X \times \ldots \times X}_{n}$ be the cartesian product of X with itself n-times. Let the norm in Y be given by $\| U \|_Y = \sqrt{\sum_{i=1}^{n} \| u_i \|_X^2}$ for $U = [u_1, \ldots, u_n] \in Y$. Consider the mapping $T: Y \to Y$ defined by $T(U) = [K_1 F_1 u, \ldots, K_n F_n u]$ where $U = [u_1, \ldots, u_n] \in Y$ and $u = \sum_{i=1}^{n} u_i$. Clearly, T is a compact continuous mapping from Y into Y . Now to complete the proof of the theorem it suffices to show, by Leray-Schauder Principle, that there is an $R > 0$ such that $(I + tT)(U) \neq 0$ for every $t \in [0,1]$ and every $U \in Y$ with $\| U \|_Y = R$, where I denotes the identity mapping on Y . Now, let $R > 0$ be such that

$$\alpha - \beta - \sqrt{\sum_{i=1}^{n} \| F_i(0) \|_{X^*}^2} / R > 0 .$$

Such an R exists since $\alpha - \beta > 0$ by assumption. We assert that $(I + tT)(U) \neq 0$ for every $t \in [0,1]$ and every U in Y with $\| U \|_Y = R$. This is obvious for $t = 0$. For $t > 0$, suppose on the contrary that there is a $U = [u_1, \ldots, u_n] \in Y$ with $\| U \|_Y = R$ in Y such that

$(I + tT)(U) = [u_1 + tK_1 F_1 u, \ldots, u_n + tK_n F_n u] = 0$ where $u = \sum_{i=1}^{n} u_i$. This, then gives that

$$0 = \sum_{i=1}^{n} (F_i u, u_i) + t \sum_{i=1}^{n} (F_i u, K_i F_i u)$$

$$\geq t\alpha \sum_{i=1}^{n} \| K_iF_iu \|_X^2 - \beta \sum_{i=1}^{n} \| u_i \|_X^2 + \sum_{i=1}^{n} (F_i(0), u_i)$$

$$\geq (\alpha - \beta - \sqrt{\sum_{i=1}^{n} \| F_i(0) \|_{X^*}^2} /R)R^2 > 0$$

a contradiction. Hence $(I + tT)(U) \neq 0$ for every $t \in [0,1]$ and every $U \in Y$ with $\|U\|_Y = R$ and so there exists a $U = [u_1, \ldots, u_n] \in Y$ such that

$0 = (I + T)(U) = [u_1 + K_1F_1u, \ldots, u_n + K_nF_nu]$ where $u = \sum_{i=1}^{n} u_i$.

It is then immediate that there is a u in X such that $u + \sum_{i=1}^{n} K_iF_iu = 0$. Hence the Theorem.

Remark 3: Theorem 2 above generalizes and also simplifies the main result of [8] since our condition $(w, K_iw) \geq \alpha \| K_iw \|_X^2$ is a proper weakening of the condition of angle-boundedness even for compact mappings (see [7]).

Remark 4: If we replace condition (3) in Theorem 2 by the condition

$$(3)' \qquad \sum_{i=1}^{n} (F_iu, u_i) \geq - \beta \sum_{i=1}^{n} \| u_i \|_X^2$$

we can then assume that $\beta \leq \alpha$ instead of $\beta < \alpha$. With this observation Theorem 2 generalizes Theorem 2.1 of [4] in the case of compact K_i's when $n > 1$.

Theorem 3. *Let Λ be a measure space with a finite measure $d\xi$. Let us suppose that we are given measurable families of compact monotone linear mappings* $\{K_\alpha : \alpha \in \Lambda\}$

from X^* into X and of bounded demi-continuous (nonlinear) mappings $\{F_\alpha : \alpha \in \Lambda\}$ from X into X^*. Suppose that there exist a constant k such that $\|K_\alpha\| \leq k$ for $\alpha \in \Lambda$ and that for each $u \in X$, $\|F_\alpha(u)\|_{X^*}$ is essentially-bounded on Λ. Let R be the mapping of $L^2(\Lambda, X)$ into X given by

$$R(u) = \int_\Lambda u(\alpha) d\xi(\alpha).$$

Suppose further that there exists an $r > 0$ such that for elements $u = \{u(\alpha)\}_{\alpha \in \Lambda}$ in $L^2(\Lambda, X)$ with $\int_\Lambda \|u(\alpha)\|_X^2 d\xi(\alpha) = r^2$ we have

$$\int_\Lambda (F_\alpha(R(u)), u(\alpha)) d\xi(\alpha) \geq 0$$

Then the mapping $T: X \longrightarrow X$ defined by

$$Tu = \int_\Lambda K_\alpha(F_\alpha(u)) d\xi(\alpha)$$

for $u \in X$ is such that the equation $u + Tu = 0$ has at least one solution u in X.

Theorem 4. Let Λ be a measure space with a finite measure $d\xi$. Let us suppose that we are given a measurable family $\{K_\alpha : \alpha \in \Lambda\}$ of compact linear mappings from X^* into X such that there exist a constant $c > 0$ such that $(w, K_\alpha w) \geq c \|K_\alpha w\|_X^2$ for every $w \in X^*$ and $\alpha \in \Lambda$. Let $\{F_\alpha : \alpha \in \Lambda\}$ be a corresponding measurable family of bounded demicontinuous (nonlinear) mappings from X into X^* such that for each $u \in X$, $\|F_\alpha u\|_{X^*}$ is essentially bounded on Λ. Let R be the mapping of $L^2(\Lambda, X)$ into X given by

$$R(u) = \int_\Lambda u(\alpha) d\xi(\alpha).$$

Suppose that there exist a constant $d > 0$ with $c > d$ such that

$$\int_{\Lambda}(F_{\alpha}(R(u)),u(\alpha))d\xi(\alpha) \geq -d\int_{\Lambda}\|u(\alpha)\|_X^2 d\xi(\alpha) + \\ + \int_{\Lambda}(F_{\alpha}(0),u(\alpha))d\xi(\alpha)$$

for each $u = \{u(\alpha)\}$ in $L^2(\Lambda, X)$.
Then the mapping $T: X \to X$ defined by

$$Tu = \int_{\Lambda} K_{\alpha}(F_{\alpha}(u))d\xi(\alpha)$$

for $u \in X$ has the property that the equation $u + Tu = 0$ has at least one solution u in X .

We omit the proofs of Theorems 3 and 4 as they are analogous to the proofs of Theorems 1 and 2 with obvious modifications.

Remark 5: Theorem 3 and 4 generalize Theorem 5.2 of [4] when the mappings K_{α}'s are compact. Also we do not need the measurability considerations as in Theorem 5.2 of [4].

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Northern Illinois University

Department of Mathematics

Dekalb, Ill. 60115

U.S.A.

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