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APPLICATIONS OF THE INDUCED MORPHISM THEOREM IN REGULAR CATEGORIES

Temple H. FAY, Conway

Abstract: Several set theoretic results, principally, due to Riguet, and a group theoretic result called Goursat's Theorem are shown to be consequences of a categorical result called the Induced Morphism Theorem (IMT). A characterization of when a regular category is exact is also derived from the IMT. The Difunctionally Induced Morphism Theorem, generalized from a result of Norris and Bednarek, is shown to be equivalent to the IMT in an exact category.

Key words: Induced Morphism Theorem, regular epimorphism, regular category, exact category, difunctional relation, congruence relation.

AMS: 18A20, 18A30 Ref. Ž.: 2.726.2

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1. Introduction. The purpose of this paper is to obtain a result called the Induced Morphism Theorem (IMT) which is a categorial generalization of a set theoretic result of Bednarek and Wallace [3] and which in turn is a consequence of a property of morphisms which characterizes the class of regular epimorphisms. Also a result of Norris and Bednarek [22] is generalized to a categorical setting and shown to be equivalent to the IMT.

Applications of the IMT are given in regular categories (terminology due to Barr [2] and Grillet [10]) and, in particular, our results subsume set theoretic results of Riguet [23, 24], and a group theoretic result called Goursat's Theorem [14, 15]. A characterization of when a regular category is exact due to Meisen [19] and independently found by the author will also be shown to follow from the IMT. This latter result is of interest in view of a theorem of Tierney (see Barr [2]) which roughly states that a category is abelian if and only if it is both additive and exact, and in view of the fact that varietal categories are precisely the exact algebraic categories (terminology due to Herrlich [11], see [12]).

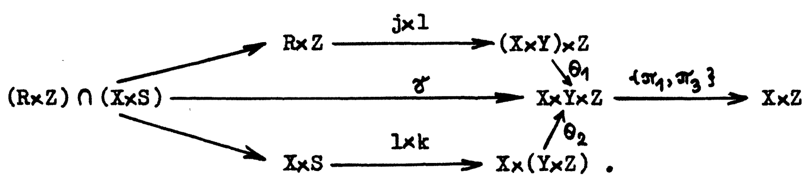
2. <u>Preliminaries</u>. If $h: X \to Y$ and $g: X \to Z$ are morphisms in a category, the unique morphism from X to the product $Y \times X$ induced by h and g is denoted $\{h,g\}$. Projection morphisms from products will be denoted by the generic symbol π with a subscript to indicate which projection; e.g., $\pi_1: X \times Y \to X$. Identity morphisms will be denoted by 1.

As usual, if (X,x) and (Y,y) are subobjects of Z, we define $(X,x) \leq (Y,y)$ if and only if there exists a morphism z such that $yz = x$. If (X,x) and (Y,y) are isomorphic as subobjects of Z, we write $(X,x) \cong (Y,y)$.

An E-M bicategory is a category C with a structure consisting of two subcategories E and M such that the morphisms in E are epimorphisms, the morphisms in M are monomorphisms, the morphisms in $E \cap M$ are precisely the isomorphisms of C, and every morphism h in C has an essentially unique factorization (e,m) , where $h = me$ and $e \in$ $\in E$ and $m \in M$. See Herrlich and Strecker [12] for a detailed development of bicategories.

Let C be a finitely complete E-M bicategory. By a relation from an object X to an object Y is meant an M-subobject (R,j) of $X \times Y$. If $X = Y$, (R,j) is called a relation on X . When there is little likelihood of confusion, the morphism j will be suppressed. Some basic categorical relation theoretic definitions and results follow. For detailed expositions, see Klein [13] or Grillet [10].

Let (R,j) be a relation from X to Y . Let (R^{-1}, j^*) be the M-monic part of the E-M factorization of $\{\pi_2, \pi_1\} j$. We call R^{-1} the inverse relation determined by R . If (S,k) is a relation from Y to Z , consider the following commutative diagram where θ_1 and θ_2 are canonical isomorphisms.



Let $(R \circ S, \alpha)$ denote the M-monic part of the E-M factorization of $\{\pi_1, \pi_2\}\gamma$; $R \circ S$ is called the composition of R with S .

Klein [13] has shown, in the case where C has pullbacks, that the composition of relations is associative if and only if the pullback of an E-epic is E-epic. In this case C is called an E-M associative bicategory.

If C is an E-M associative bicategory, then for every E-epic f and identity morphism 1, $1\times f$ and $f\times 1$ are E-epics. Thus in such a category, if f and g are E-epics, then $f\times g = (f\times 1)(1\times g)$ is an E-epic.

The inverse operation and composition respect each other via $(R\circ S)^{-1} \equiv S^{-1}\circ R^{-1}$. If R is any relation on X, R is reflexive if and only if $(\Delta_X, i_X) = (X,\{1,1\}) \leq (R,j)$. If (S,k) is a relation from X to Y, then $S\circ S^{-1}$ is reflexive if and only if $\pi_1 k$ is an E-epic. Furthermore, it follows that $\Delta_X \circ S \equiv S \equiv S\circ \Delta_Y$. One defines symmetric, transitive, and equivalence relations in an obvious fashion.

If $f: X \to Y$ is a morphism, the equalizer of $f\pi_1$ and $f\pi_2$ is denoted $(\mathrm{cong}(f), i_f)$ and is called the congruence determined by f. If (g,Z) is the coequalizer of $\pi_1 i_f$ and $\pi_2 i_f$, then there exists a morphism h such that $f = hg$ and $\mathrm{cong}(g) \equiv \mathrm{cong}(f)$. If (E,e) is a congruence on X, the coequalizer of $\pi_1 e$ and $\pi_2 e$ is denoted $(\Phi_E, X/E)$. A relation (R,j) on X is called a congruence on X if there is a morphism f with domain X such that (R,j) is the equalizer of $f\pi_1$ and $f\pi_2$ (or equivalently, $(R, \pi_1 j, \pi_2 j)$ is the pullback of f and f). Congruences are equivalence relations, but the converse need not be true.

3. __The Induced Morphism Theorem.__ The first definition of this section is a categorical property of morphisms motivated by a result which has been termed Sierpinski's Lemma [7], [20], has been atributed to Schweigert [22], called the Indu-

ced Homomorphism Theorem [6], and exploited by Aczel [1] and by Bednarek and Wallace [3].

A morphism $f: X \longrightarrow Y$ is called a Sierpinski morphism if whenever $g: X \longrightarrow Z$ is a morphism for which $\text{cong}(f) \leq \text{cong}(g)$ holds, there exists a unique morphism $h: Y \longrightarrow Z$ such that $hf = g$.

<u>Theorem 3.1</u>. In a finitely complete category having coequalizers of kernel pairs, the Sierpinski morphisms are precisely the regular epimorphisms.

<u>Proof</u>. Suppose f is a regular epimorphism, then there exist morphisms $a,b: W \longrightarrow X$ such that f is the coequalizer of a and b. Suppose g is a morphism for which $\text{cong}(f) \leq \text{cong}(g)$ holds, and consider the induced morphism $\{a,b\}: W \longrightarrow X \times X$. It follows that there exists a morphism $d: W \longrightarrow \text{cong}(f)$ such that $i_f d = \{a,b\}$. Consequently, $ga = gb$ and hence there exists a unique morphism h such that $hf = g$.

Conversely, suppose f is a Sierpinski morphism, and let $f^\#$ be the coequalizer of $\pi_1 i_f$ and $\pi_2 i_f$. It follows that $\text{cong}(f) \equiv \text{cong}(f^\#)$. There exists a morphism h such that $hf = f^\#$. It follows from the definition of coequalizer that there exists a unique morphism g such that $gf^\# = f$. One readily verifies that h and g are mutual inverses and thus f is a regular epimorphism.

The notion of Sierpinski morphism is useful in providing a simple proof to the next proposition.

<u>Proposition 3.2</u>. Let (E,e) and (F,f) be congruences on X and Y respectively and let $g: X \longrightarrow Y$ be a morphism

such that $(g\times g)e$ factors through f . Then there exists a unique morphism $h: X/E \rightarrow Y/F$ such that $h\Phi_E = \Phi_F g$.

Proof. One need only observe that Φ_E is a Sierpinski morphism and that E is "contained in" the congruence determined by $\Phi_F g$.

A finitely complete regular epi-mono associative bicategory having coequalizers of kernel pairs is called a regular category (see Barr[2] and Grillet [10]). A regular category C is exact if and only if every equivalence relation is a congruence. The categories of sets, rings, compact Hausdorff spaces, triple algebras over the category of sets for certain triples, and certain functor categories are regular categories. Varieties and abelian categories are exact. Herrlich has defined the notions of algebraic category and varietal category [11] (see [12]). Algebraic categories are regular, and varietal categories are the exact algebraic categories.

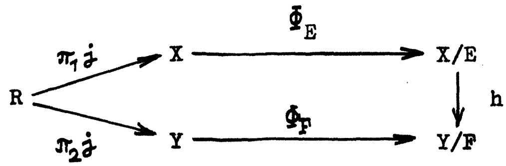
We next turn to a result noted by Bourbaki [4] in the case of the category of sets and by Bednarek and Wallace [3] in the case of the category of compact Hausdorff spaces.

Proposition 3.3. In a regular category, if E and F are congruences on X and Y respectively with the congruence determined by $\Phi_E \times \Phi_F$ denoted by $E * F$, then $(\Phi_E \times \Phi_F, E\times Y/F)$ and $(\Phi_{E*F}, X\times Y/E*F)$ are isomorphic as quotients.

Proof. Observe that Φ_{E*F} and $\Phi_E \times \Phi_F$ are Sierpinski morphisms.

Theorem 3.4. The Induced Morphism Theorem (IMT) Let (R,j) be a relation from X to Y in a regular category for which $\pi_1 j$ is a regular epimorphism. If (E,e) and (F,f)

are congruences on X and Y respectively such that $R^{-1} \circ (E \circ R) \leq F$, then there exists a unique morphism h such that the following diagram is commutative.



<u>Proof</u>. (Sketch) Define $\Phi = \Phi_E \pi_1 j$ and $\psi = \Phi_F \pi_2 j$. It follows that there exists a (unique) morphism g such that $(\pi_1 j \times \pi_1 j) i_\Phi = eg$. From this it is straightforward to verify that $\{\pi_1 j\, \pi_1 i_\Phi , \pi_2 j\, \pi_2 i_\Phi\} : \mathrm{cong}(\Phi) \longrightarrow X \times Y$ factors through $E \circ R$. This implies that $(\pi_2 j \times \pi_2 j) i_\Phi$ factors through $R^{-1} \circ (E \circ R)$ and consequently, through f . Thus $\mathrm{cong}(\Phi) \leq \mathrm{cong}(\psi)$. Since Φ is a Sierpinski morphism, there exists a unique morphism h such that $h\Phi = \psi$.

This proof does not depend on the associativity of composition of relations, and is true in the category of topological spaces where the pullback of a quotient map (= regular epic) is not necessarily a quotient map.

This theorem is generalized from a result of Bednarek and Wallace [3] called the Induced Function Theorem (IFT) and which Norris [21] has recently used with some success. Martin [17] has shown how isomorphism theorems in algebra may be deduced from the IFT, and more recently, Norris and Bednarek [22] and Martin [18] have demonstrated some equivalent theorems.

<u>Corollary 3.5</u>. If (R,j) is a difunctional relation $(R \circ (R^{-1} \circ R) \equiv R \equiv (R \circ R^{-1}) \circ R)$ from X to Y in an exact

category and if $\pi_1 j$ and $\pi_2 j$ are regular epimorphisms, then there exists an isomorphism h such that the following diagram is a pushout square.

$$\begin{array}{ccc} R & \xrightarrow{\pi_1 j} & X \\ & & \downarrow \Phi_{R\circ R^{-1}} \\ \pi_2 j \downarrow & & X/R\circ R^{-1} \\ & & \downarrow h \\ Y & \xrightarrow{\Phi_{R^{-1}\circ R}} & Y/R^{-1}\circ R \end{array}$$

<u>Proof</u>. Since R is difunctional, $R\circ R^{-1}$ and $R^{-1}\circ R$ are equivalence relations, hence are congruences. In view of the IMT, it need only be observed that $R^{-1}\circ((R\circ R^{-1})\circ R) \leq \leq R^{-1}$ and $R\circ((R^{-1}\circ R)\circ R^{-1}) \leq R\circ R^{-1}$. These follow immediately since R is difunctional.

To see the diagram is a pushout square, suppose $a: X \rightarrow \rightarrow W$ and $b: Y \rightarrow W$ are morphisms for which $a\pi_1 j = b\pi_2 j$. Let $(R^{-1}\circ R, k)$ be the indicated composition; it follows that $b\pi_1 k = b\pi_2 k$. Consequently, there exists a morphism $s: : Y/R^{-1}\circ R \rightarrow W$ such that $b = s\Phi_{R^{-1}\circ R}$. We see that $sh\Phi_{R\circ R^{-1}}\pi_1 j = s\Phi_{R^{-1}\circ R}\pi_2 j = b\pi_2 j = a\pi_1 j$; $\pi_1 j$ is an epimorphism, hence $sh\Phi_{R\circ R^{-1}} = a$. Thus the diagram is a pushout square.

A recent result of Norris and Bednarek [22] called the Difunctionally Induced Function Theorem (DIFT) is shown to be equivalent to the IFT. We now generalize the DIFT to a categorical setting and, being motivated by the proof given in [22], show that what we call the Difunctionally Induced Morphism Theorem (DIMT) is equivalent to the IMT in an exact category.

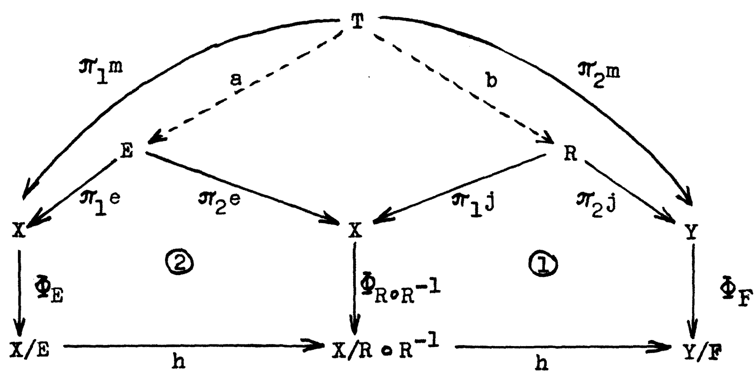
<u>Theorem 3.6</u>. <u>The Difunctionally Induced Morphism Theorem</u> in an exact category, let (R,j) and (S^{-1},k^{*}) be difunctional relations from X to Y such that $\pi_1 j$ and $\pi_2 k^{*}$ are regular epimorphisms. If $R^{-1} \circ R \leq S \circ S^{-1}$, then there exists a unique morphism h making the following diagram commute.

$$\begin{array}{ccccc} & \overset{\pi_1 j}{\nearrow} & X & \xrightarrow{\Phi_{R\circ R^{-1}}} & X/R\circ R^{-1} \\ R & & & & \downarrow h \\ & \underset{\pi_2 j}{\searrow} & Y & \xrightarrow{\Phi_{S\circ S^{-1}}} & Y/S\circ S^{-1} \end{array}$$

<u>Proof</u>. By hypothesis, $R^{-1} \circ ((R \circ R^{-1}) \circ R) = R^{-1} \circ R \leq S \circ S^{-1}$. Thus the result follows from the IMT. Note associativity of composition of relations is not needed here.

<u>Theorem 3.7</u>. In an exact category, the IMT is equivalent to the DIMT.

<u>Proof</u>. Associativity will be used throughout this proof. In view of Theorem 3.6, it need only be observed how the IMT is deduced from the DIMT. Let (T,m) be a relation from X to Y such that $\pi_1 m$ is a regular epimorphism. Let (E,e) and (F,f) be congruences on X and Y respectively and suppose $T^{-1} \circ (E \circ T) \leq F$. Let $(R,j) = (E \circ T \circ F, j)$ and let $S = R^{-1}$. Observe R and S are difunctional. Since $\pi_1 m$ is a regular epimorphism and $T \leq R$, it follows that $\pi_1 j$ and $\pi_2 j$ are also regular epimorphisms.



It will be shown that the above diagram is commutative. Observe that $F = S \circ S^{-1}$ and that $R^{-1} \circ R = S \circ S^{-1}$. Thus the DIMT implies there exists a morphism h_1 such that "pentagon ① " commutes.

Next observe that $\pi_1 e$ is a regular epimorphism and that $E \cong R \circ R^{-1}$. Thus the DIMT implies there exists a morphism h_2 such that "pentagon ② " commutes.

Since $T \leq R$, there exists a morphism b such that $jb = m$. Furthermore, $(T, \{\pi_1 m, \pi_1 m\}) \leq (\Delta_X, i_X) \leq (E, e)$ so that there exists a morphism a such that $ea = \{\pi_1 m, \pi_1 m\}$. Thus the outer "pentagon" commutes.

In vies of the applications to topological algebra of the IFT investigated by Bednarek and Wallace [3] and Norris [21] , a remark concerning the IMT is in order. The setting of an exact category (or of a regular category) seems to be the proper setting for doing most algebraic constructions (e.g., manipulating congruences, forming quotients, etc.), but the pullback axiom seems to limit potential applications in topological settings (except for the category of compact Hausdorff spaces which is a variety, and in which all of the above theorems hold).

4. <u>Difunctional Relations</u>. Let C be a finitely complete E-M bicategory and let (R,j) be a relation from X to Y. Denote the E-M factorizations of $\pi_1 j$ and $\pi_2 j$ by (τ_1, j_1) and (τ_2, j_2) respectively. The domain of j_1 is denoted RY and the domain of j_2 is denoted XR. Essentially, $RY \times XR$ is the "smallest" rectangular relation "containing" R. If (S,k) is a relation from X to Y such that $R \leq S$, then $RY \leq SY$ and $XR \leq XS$. It also follows that $X(R \circ R^{-1}) \equiv RY$, $(R^{-1} \circ R)Y \equiv XR$, and $RY \equiv YR^{-1}$. The proof of the next lemma is omitted.

<u>Lemma 4.1</u>. If C is as above and (R,j) is a relation from X to Y, then:

(1) $\pi_1 j$ is an E-epic if and only if $RY \equiv X$.

(2) $\pi_1 j$ is an E-epic if and only if $R \circ R^{-1}$ is reflexive on X.

(3) $R \circ \Delta_{XR} \equiv R \equiv \Delta_{RY} \circ R$ where $(\Delta_{RY}, (j_1 \times j_1) i_{RY})$ and $(\Delta_{XR}, (j_2 \times j_2) i_{XR})$ are considered as relations on X and Y respectively.

If (R,j) is a relation from X to Y, then $(R, \{\tau_1, \tau_2\})$ is a relation from RY to XR, called the canonical embedding of R into $RY \times XR$. Denote $\{\tau_1, \tau_2\}$ by Φ. Note $(j_1 \times j_2)\Phi = j$. Let (S,k) be a relation from Y to Z such that $XR \equiv SZ$ and let (S,ψ) denote the canonical embedding of S into $XR \times YS$.

Theorem 4.2. If $(R \circ S, m)$ denotes the composition of (R, j) with (S, k), and if $(\widetilde{R \circ S}, \tilde{m})$ denotes the composition of (R, Φ) with (S, ψ), then $(R \circ S, m) \equiv (\widetilde{R \circ S}, (j_1 \times k_2)\tilde{m})$.

Proof. Consider the intersections $(A = (R \times Z) \cap (X \times S), \alpha)$ and $(B = (R \times YS) \cap (RY \times S), \beta)$. It follows that $(A, \alpha) \equiv (B, (j_1 \times k_2)\beta)$. The theorem results from the essential uniqueness of the E-M factorization.

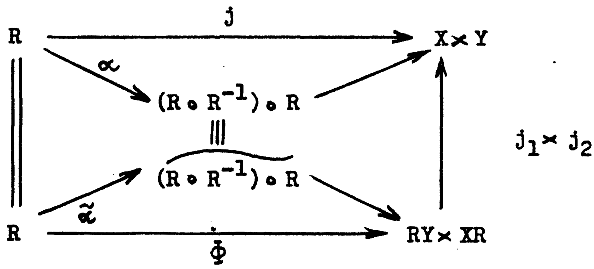
Corollary 4.3. (R, j) is a symmetric, transitive relation on X if and only if (R, Φ) is an equivalence relation on RY.

A relation (R, j) from X to Y is called difunctional if and only if $R \circ (R^{-1} \circ R) \leq R$ and $(R \circ R^{-1}) \circ R \leq R$. Note there is no assumption of associativity for composition of relations assumed here. If R is any relation, it follows from Lemma 4.1 that $R \equiv R \circ \Delta_{XR} \leq R \circ (R^{-1} \circ R)$ and $R \equiv \Delta_{RY} \circ R \leq (R \circ R^{-1}) \circ R$. Thus R is difunctional if and only if $R \circ (R^{-1} \circ R) \equiv R \equiv (R \circ R^{-1}) \circ R$.

The term difunctional is due to Riguet [23]. MacLane [16], and Lambek [14, 15] have shown the usefulness of the concept. Meisen [19] has studied difunctional relations in E-M bicategories as has Klein [13] (under the term Von Neumann regular) and Grillet [10] notes that all relations are difunctional in an abelian category. This author has shown all relations are difunctional in any additive category [8].

Corollary 4.4. (R, j) is difunctional if and only if (R, Φ) is also.

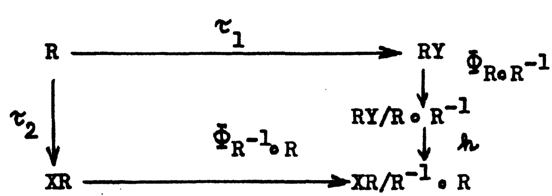
Proof. The following diagram commutes where all the morphisms are M-monomorphisms.



To avoid confusion we use the notation $\widetilde{(R \circ R^{-1}) \circ R}$ to denote the indicated compositions of (R, Φ) and (R^{-1}, Φ^*). It follows that α is an isomorphism if and only if $\tilde{\alpha}$ is an isomorphism.

The next result appears in [23] in a set theoretic setting. If C is the category of groups, this result has been called Goursat's Theorem which is useful in constructing connecting homomorphisms in homological algebra, and in giving elegant proofs for the Zassenhaus Lemma and the Jordan Hölder Schreier Refinement Theorem [14, 15].

Corollary 4.5. If C is an exact category and (R, j) is a difunctional relation from X to Y, then there exists an isomorphism h such that the following diagram is a push-out square (cf. Corollary 3.5).



Proof. By the previous corollary and Corollary 3.5, the result is immediate.

This last result plays a central role in providing a relation theoretic characterization of when a regular category is exact.

A relation (R,j) from X to Y is called a pullback relation if and only if there exist morphisms g and h such that $(R, \pi_1 j, \pi_2 j)$ is the pullback of g and h. Thus congruences are pullback relations; in fact, a relation is a congruence if and only if it is both reflexive and a pullback relation.

Theorem 4.6. Let C be a finitely complete E-M bicategory; if (R,j) is a pullback relation, then (R,j) is difunctional.

Proof. Consider the indicated compositions $(R \circ R^{-1}, b)$ and $((R \circ R^{-1}) \circ R, a)$. Since R is a pullback relation, there exist morphisms $f: X \to Z$ and $g: Y \to Z$ such that (R,j) is the equalizer of $f\pi_1$ and $g\pi_2$. It follows that $f\pi_1 b = f\pi_2 b$. Using this fact, one demonstrates that $f\pi_1 a = g\pi_2 a$, which implies $(R \circ R^{-1}) \circ R \leq R$. Similarly one shows $R \circ (R^{-1} \circ R) \leq R$.

Let $f: X \to Y$ be a morphism. The morphism $\{1,f\} : X \to X \times Y$ is a section (coretract) hence, $(X,\{1,f\})$ is a relation in any E-M bicategorical structure on C. This relation is called the graph determined by f. For the sake of convenience, we denote $(X, \{1,f\})$ by $(G_f, \hat{f})$. Observe $(G_f^{-1}, \hat{f}^*) \equiv (X,\{f,1\})$. Klein [13] and Grillet [10] have noted graphs are difunctional.

The proof of the next lemma is straightforward and is omitted.

Lemma 4.7. (a) If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are morphisms, then $G_f \circ G_g \equiv G_{gf}$.

(b) If $f: X \longrightarrow Y$ and $g: Z \longrightarrow Y$ are morphisms, then the pullback of f and g is $G_f \circ G_g^{-1}$. In particular, $\mathrm{cong}(f) \equiv G_f \circ G_f^{-1}$.

The next result is due to Meisen [19] and was independently found by the author.

Theorem 4.8. If C is an exact category, then every difunctional relation has the property that its canonical embedding is a pullback relation.

Proof. Let (R,j) be a difunctional relation from X to Y . There is no loss of generality to assume $X \equiv RY$ and $Y \equiv XR$. We have seen that the following diagram commutes.

$$\begin{array}{ccc}
R & \xrightarrow{\ \pi_1 j\ } & X \\
 & & \downarrow{\scriptstyle \Phi_{R\circ R^{-1}} = f} \\
\downarrow{\scriptstyle \pi_2 j} & & X/R\circ R^{-1} \\
 & & \downarrow{\scriptstyle h} \\
Y & \xrightarrow{\ \Phi_{R^{-1}\circ R} = g\ } & Y/R^{-1}\circ R
\end{array}$$

In view of the previous lemma, the pullback of hf and g is $G_{hf} \circ G_g^{-1}$. We will show that $R \equiv G_f \circ G_h \circ G_g^{-1}$. Since the diagram commutes, $R \leq G_f \circ G_h \circ G_g^{-1}$.

It follows that $(R, \{f\pi_1 j, g\pi_2 j\}) \leq (G_f^{-1} \circ R \circ G_g, b)$. Since $f\pi_1 j$ is a regular epimorphism, $(f\pi_1 j, \hat{h})$ is the regular epi-mono factorization of $\{f\pi_1 j, g\pi_2 j\}$ and consequently, $(G_h, \hat{h}) \leq (G_f^{-1} \circ R \circ G_g, b)$. Computing, $G_{hf} \circ G_g^{-1} \equiv G_f \circ G_h \circ G_g^{-1} \leq G_f \circ G_f^{-1} \circ R \circ G_g \circ G_g^{-1} \equiv (R \circ R^{-1}) \circ R \circ (R^{-1} \circ R) \equiv R$.

Corollary 4.9. If C is a regular category, C is exact if and only if every difunctional relation (R,j) has the property that its canonical embedding (R,Φ) is a pull-back relation.

Conditions concerning commuting congruences equivalent to a regular category being exact are given by Burgess and Caicedo [5] and by this author [9].

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