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TOLERANCE RELATIONS ON SEMILATTICES

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Abstract: A tolerance compatible with an algebra is defined similarly as a congruence, only the transitivity is not required. This paper contains some results on tolerances compatible with a semilattice.

Key words: Semilattice, tolerance.

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This paper continues the study of tolerance relations on algebras which was begun in [2],[3] and [4]. The concept of tolerance was introduced by E.C. Zeeman [1].

A tolerance relation is a binary relation on some set which is reflexive and symmetric. If $\mathfrak{A} = \langle A, \mathcal{F} \rangle$ is some algebra (A denotes the set of elements of $\mathfrak{A}$ and $\mathcal{F}$ denotes the set of its operations), and ξ is some tolerance on A , we say that ξ is compatible with $\mathfrak{A}$ if and only if the following condition is satisfied: If $f \in \mathcal{F}$ is an n-ary operation, where n is a positive integer, and $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ are elements of A such that $(x_i, y_i) \in \xi$ for $i = 1, \ldots, n$, then

$$(f(x_1, \ldots, x_n) \ , \ f(y_1, \ldots, y_n)) \in \xi \ .$$

Here we shall study tolerances on semilattices. If a semilattice is not considered as a part of a lattice, the operation in it is called multiplication and denoted by $\circ$, its result is called product. The ordering on a semilattice S is defined so that for $a \in S$, $b \in S$ we have $a \leqq b$ if and only if $a \circ b = b$. If a semilattice is considered as a part of a lattice, we use the signs $\vee$ and $\wedge$ for the lattice operations and call them join and meet.

Thus, if S is a semilattice and ξ is a tolerance on the set of elements of S , then ξ is compatible with S if and only if for any $x_1 \in S$, $x_2 \in S$, $y_1 \in S$, $y_2 \in S$ such that $(x_1,y_1) \in \xi$, $(x_2,y_2) \in \xi$ we have $(x_1 \circ x_2 , y_1 \circ y_2) \in \xi$.

__Theorem 1.__ Let S be a semilattice, let ξ be a tolerance compatible with S . Let $x \in S$. The set $S(x) = \{y \in S \mid (x,y) \in \xi\}$ is a subsemilattice of S . Moreover, if $S(x)$ has the greatest element $M(x)$ for each $x \in S$, then the mapping M which assigns $M(x)$ to x for each $x \in S$ is an isotone mapping of S into itself.

__Proof.__ A semilattice is a commutative semigroup in which all elements are idempotents. Thus $\{x\}$ for each $x \in S$ is a subsemilattice of S and according to Theorem 4 from [2] also $S(x)$ is a subsemilattice of S . The assertion for $M(x)$ is proved analogously to the proof of Theorem 12 from [2] ; that theorem is proved for lattices, but in its proof no meets are used.

Now if $a \in S$, $b \in S$, $a \leqq b$, then the interval $\langle a,b \rangle$ is by definition the set $\{x \in S \mid a \leqq x \leqq b\}$.

__Theorem 2.__ Let S be a semilattice, let ξ be a tolerance compatible with S . Let $x \in S$, $y \in S$, $(x,y) \in \xi$. Then $(x \circ y, z) \in \xi$ for each $z \in \langle x, x \circ y \rangle \cup \langle y, x \circ y \rangle$.

<u>Proof</u>. Let $z \in \langle x, x \circ y \rangle \cup \langle y, x \circ y \rangle$. We have $(x,y) \in \xi$, $(z,z) \in \xi$, therefore $(x \circ z, y \circ z) \in \xi$. Evidently, $\langle x, x \circ y \rangle \cup \langle y, x \circ y \rangle = \{x \circ y\}$, thus $y \circ z = x \circ y$ for each $z \in \langle x, x \circ y \rangle$ and $x \circ z = x \circ y$ for each $z \in \langle y, x \circ y \rangle$. Thus if $z \in \langle x, x \circ y \rangle$, we have $z \geqq x$, thus $x \circ z = z$ and further $y \circ z = x \circ y$; this means $\xi \ni (x \circ z, y \circ z) = (z, x \circ y)$. If $z \in \langle y, x \circ y \rangle$ then $x \circ z = x \circ y$, $y \circ z = z$ and we have again $(z, x \circ y) \in \xi$.

This is a substantial difference in comparison with the case of lattices [4]. In the case of semilattices it is not necessary that any two elements of $\langle x, x \circ y \rangle \cup \langle y, x \circ y \rangle$ should be in ξ . For example, let C_1 , C_2 be two disjoint chains of the cardinality greater than one with the least elements c_1 , c_2 respectively, let 0 be an element which does not belong to $C_1 \cup C_2$. Put $S = C_1 \cup C_2 \cup \{0\}$ and define the ordering in S so that $x \leqq y$ if and only if either both x and y are in C_1 and $x \leqq y$ holds in C_1 , or both x and y are in C_2 and $x \leqq y$ holds in C_2 , or $y = 0$ and x is an arbitrary element of S . Let ξ be a tolerance relation on S consisting of the pairs (c_1,c_2) , (c_2,c_1) and the pairs (x,x) , $(x,0)$, $(0,x)$ for each $x \in S$. The tolerance ξ is compatible with S .

<u>Theorem 3</u>. Let S be a semilattice with more than two elements. Then there exists a tolerance ξ compatible with

S which is not a congruence.

Proof. At first let S be a chain. Let a be an element of S which is neither the greatest, nor the least one; such an element exists, because S has at least three elements. Let ξ consist of all pairs (x,y), where either simultaneously $x \geqq a$, $y \geqq a$, or simultaneously $x \leqq a$, $y \leqq a$. Let $(x_1,y_1) \in \xi$, $(x_2,y_2) \in \xi$. If at least one of these pairs has the property that both elements are greater than or equal to a, then $x_1 \circ x_2 \geqq a$, $y_1 \circ y_2 \geqq a$ and $(x_1 \circ x_2, y_1 \circ y_2) \in \xi$. If $x_1 \leqq a$, $x_2 \leqq a$, $y_1 \leqq a$, $y_2 \leqq a$, then also $x_1 \circ x_2 \leqq a$, $y_1 \circ y_2 \leqq a$ and again $(x_1 \circ x_2, y_1 \circ y_2) \in \xi$. Thus ξ is compatible with S. Now let $b < a < c$. We have $(b,a) \in \xi$, $(a,c) \in \xi$, but $(b,c) \notin \xi$, thus ξ is not transitive and it is not a congruence.

Now suppose that S is not a chain. Let a, b be two incomparable elements of S. Take a tolerance ξ consisting of the pairs (x,x), $(y, a \circ b)$, $(a \circ b, y)$, $(y \circ x, a \circ b \circ x)$, $(a \circ b \circ x, y \circ x)$ for each $x \in S$, $y \in \langle a, a \circ b \rangle \cup \langle b, a \circ b \rangle$. This is evidently a tolerance compatible with S. We have $(a, a \circ b) \in \xi$, $(a \circ b, b) \in \xi$; but $(a,b) \notin \xi$, because $a \neq b$ and none of the elements a, b can be equal to $a \circ b$ or $a \circ b \circ x$ for some $x \in S$.

Now we shall consider upper and lower semilattices of a lattice.

Theorem 4. Let L be a lattice with more than two elements, let $L(\vee)$ be the upper semilattice of L, let

$L(\wedge)$ be the lower semilattice of L. Then there exist tolerances ξ, ξ' on L such that ξ is compatible with $L(\vee)$, ξ' is compatible with $L(\wedge)$, but none of them is compatible with L.

<u>Proof</u>. Suppose that L is not a chain. Then there exist elements a, b of L which are incomparable. We construct the tolerance ξ analogously as in the proof of Theorem 3; the tolerance ξ is compatible with $L(\vee)$. Suppose that it is compatible with L. From $(a, a \vee b) \in \xi$, $(a \vee b, b) \in \xi$ we obtain $(a \wedge (a \vee b), (a \vee b) \wedge b) = (a,b) \in \xi$, which is a contradiction. If L is a chain, let a be an element of L to which at least two elements b, c exist such that $b < c < a$. Then ξ consists of the pairs (x,x), (a,y), (y,a) for all $x \in S$ and all $y \leqq a$. Let x_1, x_2, y_1, y_2 be elements of L, $(x_1,y_1) \in \xi$, $(x_2,y_2) \in \xi$. If $x_1 = y_1$, $x_2 = y_2$, then $x_1 \vee x_2 = y_1 \vee y_2$ and $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$. If $x_1 = a$, $y_1 \leqq a$, $x_2 = y_2 = a$, then $x_1 \vee x_2 = x_2$, $y_1 \vee y_2 = y_2$ and $(x_1 \vee x_2, y_1 \vee y_2) = (x_2, y_2) \in \xi$. If $x_1 = a$, $y_1 \leqq a$, $x_2 = y_2 \leqq a$, then $x_1 \vee x_2 = a$, $y_1 \vee y_2 \leqq a$, thus again $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$. If $x_1 = a$, $y_1 \leqq a$, $x_2 = a$, $y_2 \leqq a$ or $x_1 = a$, $y_1 \leqq a$, $x_2 \leqq a$, $y_2 = a$, then $x_1 \vee x_2 = a$, $y_1 \vee y_2 \leqq a$ and $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$. All other cases are obtained from some of these cases by changing the notation. Thus ξ is compatible with $L(\vee)$. Now let $c < d < a$. We have $(c,a) \in \xi$, $(a,d) \in \xi$, but $(c,d) = (c \wedge a, a \wedge d) \notin \xi$; the tolerance ξ is not compatible with L. The construction of ξ' is dual to this

construction.

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