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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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EXTENSION OF SEQUENTIALLY CONTINUOUS MAPPINGS

Roman FRIČ, Žilina

Abstract: A.D. Tajmanov proved in [7] a necessary and sufficient condition for a continuous mapping of a dense subspace of a T_1 topological space into a compact Hausdorff space to be continuously extended onto the whole space. We prove a similar result for convergence, resp. sequential, spaces.

Key words: Convergence space, sequential space, extension of a (sequentially) continuous mapping, sequentially complete convergence, resp. sequential, space.

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The reader is asked to refer for the background material on closure spaces to [1], convergence spaces to [5], and sequential spaces to [2]. The convergence of sequences in sequential spaces is briefly discussed in [3]. Throughout the paper we shall always assume that a closure space has unique sequential limits and hence it is a T_1 space. We employ the symbol $f: (P,u) \longrightarrow (Q,v)$ to denote a continuous mapping of a closure space (P,u) into a closure space (Q,v). If (Q,v) is a convergence space or a sequential space, then f is continuous iff it is sequentially continuous. Recall (cf. [4]), that a closure space (P,u) is called sequentially regular if the convergence of sequences in (P,u) is projectively generated by C(P), i.e. $x = \lim x_n$ iff $f(x) = \lim f(x_n)$ for each $f \in C(P)$. A sequentially regular convergence, resp. sequential, space is called sequentially complete if it is closed in each sequentially regular convergence, resp. sequential, space in which it is C-embedded. A sequentially regular convergence space (L,λ) is a sequential envelope of itself iff (L,λ) is sequentially complete [1].

Our starting point is the above mentioned Tajmanov's result:

Theorem 1. Let X be a dense subset of a topological space (P,u) and (Q,v) a compact topological space. Then $f\colon (X, u/_X) \longrightarrow (Q,v)$ can be extended to $\bar{f}\colon (P,u) \longrightarrow (Q,v)$ iff the following condition is satisfied:

(1) $A, B \subset Q$, $vA \cap vB = \emptyset$ implies $(uf^{\leftarrow}[A]) \cap \cap (uf^{\leftarrow}[B]) = \emptyset$.

Lemma 2. Let (P,u) and (Q,v) be topological spaces. Let f be a mapping of a subset $X \subset P$ into Q such that the condition (1) is satisfied. Then $f\colon (X,u/_X) \longrightarrow (Q,v)$.

The straightforward proof is omitted.

As a simple corollary of Lemma 2 in [6] we have

Lemma 3. Let (L,λ) be a convergence space, $X \subset L$, and $x \in \lambda^{\omega_1} X$. Then there is a countable set $S \subset X$ such that $x \in \lambda^{\omega_1} S$.

Theorem 4. Let (L,λ) be a convergence space, $X \subset L$, $\lambda^{\omega_1} X = L$, and (M,μ) a sequentially complete sequentially

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1) In [5] the term $\mathcal{L}$-complete is used instead.

regular convergence space. Let $f: (X, \lambda/_X) \longrightarrow (M, \mu)$. Then f can be extended to $\overline{f}: (L, \lambda) \longrightarrow (M, \mu)$ iff the following condition is satisfied:

(2) $S_1, S_2 \subset X$, $\operatorname{card} S_1 \leq \aleph_0$, $(\mu^{\omega_1} f[S_1]) \cap \cap (\mu^{\omega_1} f[S_2]) = \emptyset$ implies $\lambda^{\omega_1} S_1 \cap \lambda^{\omega_1} S_2 = \emptyset$.

Proof. (2) is necessary. If $\overline{f}: (L, \lambda) \longrightarrow (M, \mu)$, $\overline{f}/_X = f$, then it follows from 16 B.4 in [1] that $\overline{f}: (L, \lambda^{\omega_1}) \longrightarrow (M, \mu^{\omega_1})$ and (2) is obvious.

(2) is sufficient. Let μ be the completely regular modification of μ, (Q,v) the Čech-Stone compactification of (M, μ), and (Q, ν) the convergence space associated with (Q,v). From Theorem 11 in [5] it follows that $\mu = \nu/_M$ and since (M, μ) is sequentially complete, we have $\nu M = M$. Plainly $f: (X, \lambda/_X) \longrightarrow (Q, \nu)$ and $f: (X, \lambda/_X) \longrightarrow (Q,v)$. Denote by $P = L$ and $u = \lambda^{\omega_1}$. Using (2) and Lemma 3 it can be easily proved that the condition (1) is satisfied. It follows from Lemma 2 that $f: (X, u/_X) \longrightarrow (Q,v)$ and hence, by Theorem 1, f can be extended to $\overline{f}: (P, u) \longrightarrow (Q,v)$. From 35 C.9 in [1] it follows that $\overline{f}: (L, \lambda) \longrightarrow (Q, \nu)$. Since $f[X] \subset M$ and $\nu M = M$, we have $\overline{f}[L] \subset M$. Thus $\overline{f}: : (L, \lambda) \longrightarrow (M, \mu)$ and the proof is finished.

Corollary 5. Let (L, λ^{ω_1}) be a sequential space, $X \subset L$, $\lambda^{\omega_1} X = L$, and (M, μ^{ω_1}) a sequentially complete sequentially regular sequential space. Then $f: (X, \lambda^{\omega_1}/_X) \longrightarrow (M, \mu^{\omega_1})$ can be extended to $\overline{f}: (L, \lambda^{\omega_1}) \longrightarrow (M, \mu^{\omega_1})$ iff the condition (2) is satisfied.

Corollary 6. Let (L, λ) be a convergence space, $X \subset L$,

$\lambda^{\omega_1} X = L$, and $C_0 \subset C(L)$. Then $(X, \lambda/_X)$, resp. $(X, \lambda^{\omega_1}/_X)$, can be C_0 - embedded into (L, λ) , resp. (L, λ^{ω_1}) , iff the following condition is satisfied:

(3) $S_1 , S_2 \subset X$, card $S_i \leq \aleph_0$, $\overline{f[S_1]} \cap \overline{f[S_2]} = \emptyset$ for some $f \in C_0$ implies $\lambda^{\omega_1} S_1 \cap \lambda^{\omega_1} S_2 = \emptyset$.

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