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ON SURFACES WITH CONSTANT MEAN CURVATURE

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Abstract: A global characterization of surfaces with constant mean curvature.

Key words: Surface, mean curvature, integral formula.

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A. Švec [2] used higher derivatives of the mean and Gauss curvature in order to characterize the sphere; he proved his results by means of the maximum principle. In what follows, I use an integral formula to prove a theorem of a similar type.

Theorem. *Let $M \subset E^3$ be a surface of class C^∞ with positive Gauss curvature, let ∂M be its boundary. Suppose that there is, on M, a couple of orthogonal unit tangent vector fields V_1, V_2 such that*

(1) $$V_1 V_1 H = 0 \ , \quad V_2 H = 0 \quad \text{on} \quad M \ ,$$

H *being the mean curvature of* M. *Further, suppose*

(2) $$V_1 H = 0 \ , \quad V_2 H = 0 \quad \text{on} \quad \partial M \ .$$

Then M *has constant mean curvature.*

Proof. 1. On M, consider fields of orthonormal frames $\{m, v_1, v_2, v_3\}$ with $v_1, v_2 \in T_m(M)$, $m \in M$. Then

(3) $$dm = \omega^1 v_1 + \omega^2 v_2 ,$$

(4) $$dv_1 = \omega^2 v_2 + \omega^3 v_3 , \; dv_2 = -\omega^2 v_1 + \omega^3 v_3 , \; dv_3 = -\omega_1^3 v_1 - \omega_2^3 v_2 ; \; \omega_1^3 = a\omega^1 + b\omega^2, \; \omega_2^3 = b\omega^1 + c\omega^2 ;$$

(5) $$da - 2b\omega_1^2 = \alpha\omega^1 + \beta\omega^2 ,$$
$$db + (a - c)\omega_1^2 = \beta\omega^1 + \gamma\omega^2 ,$$
$$dc + 2b\omega_1^2 = \gamma\omega^1 + \delta\omega^2 ;$$

(6) $$d\alpha - 3\beta\omega_1^2 = A\omega^1 + (B - bK)\omega^2 ,$$
$$d\beta + (\alpha - 2\gamma)\omega_1^2 = (B + bK)\omega^1 + (C + aK)\omega^2 ,$$
$$d\gamma + (2\beta - \delta)\omega_1^2 = (C + cK)\omega^1 + (D + bK)\omega^2 ,$$
$$d\delta + 3\gamma\omega_1^2 = (D - bK)\omega^1 + E\omega^2 ,$$

see [2]. Let $\{m, w_1, w_2, w_3\}$ be another field of moving frames; let

(7) $$v_1 = \varepsilon_1 \cos\varphi . w_1 - \varepsilon_1 \sin\varphi . w_2 , \; v_2 = \sin\varphi . w_1 + \cos\varphi . w_2 ,$$
$$v_3 = \varepsilon_2 w_3 ; \quad \varepsilon_1^2 = \varepsilon_2^2 = 1 .$$

Write

$$(8) \qquad dm = \tau^1 w_1 + \tau^2 w_2 ,$$

$$dw_1 = \tau_1^2 w_2 + \tau_1^3 w_3 , \; dw_2 = -\tau_1^2 w_1 + \tau_2^3 w_3 , \; dw_3 = -\tau_1^3 w_1 - \tau_2^3 w_2 ,$$

and denote by * the expressions associated to $\{m, w_1, w_2, w_3\}$. We get

$$(9) \qquad \tau^1 = \varepsilon_1 \cos\varphi . \omega^1 + \sin\varphi . \omega^2 , \quad \tau^2 = -\varepsilon_1 \sin\varphi . \omega^1 + \cos\varphi . \omega^2$$

and

$$(10) \qquad \alpha^* + \gamma^* = \varepsilon_1 \varepsilon_2 \cos\varphi . (\alpha + \gamma) + \varepsilon_2 \sin\varphi . (\beta + \delta) ,$$

$$\beta^* + \delta^* = -\varepsilon_1 \varepsilon_2 \sin\varphi . (\alpha + \gamma) + \varepsilon_2 \cos\varphi . (\beta + \delta) ;$$

see [1]. The mean and Gauss curvatures are defined by

$$(11) \qquad 2H = a + c , \quad K = ac - b^2$$

resp.; we have

$$(12) \qquad H^* = \varepsilon_2 H , \quad K^* = K .$$

2. Let us deduce an integral formula. Let

$$(13) \qquad \varphi = R_1 \omega^1 + R_2 \omega^2$$

be a 1-form on M. From

$$R_1 \omega^1 + R_2 \omega^2 = R_1^* \tau^1 + R_2^* \tau^2$$

and (9), we deduce

$$(14) \qquad R_1^* = \varepsilon_1 \cos\varphi . R_1 + \sin\varphi . R_2 , \; R_2^* = -\varepsilon_1 \sin\varphi . R_1 + \cos\varphi . R_2 .$$

The covariant derivatives R_{ij} of R_i with respect to ω^1, ω^2 be defined by

$$(15) \qquad dR_1 - R_2\omega_1^2 = R_{11}\omega^1 + R_{12}\omega^2 ,$$
$$dR_2 + R_1\omega_1^2 = R_{21}\omega^1 + R_{22}\omega^2 .$$

Similarly, define the covariant derivatives R^*_{ij} of R^*_i with respect to τ^1, τ^2. Using (14),(9) and

$$(16) \qquad \tau_1^2 - d\varphi = \varepsilon_1 \omega_1^2 ;$$

see [1], we get

$$\varepsilon_1 \cos\varphi . R^*_{11} - \varepsilon_1 \sin\varphi . R^*_{12} = \varepsilon_1 \cos\varphi . R_{11} + \sin\varphi . R_{21} ,$$
$$\sin\varphi . R^*_{11} + \cos\varphi . R^*_{12} = \varepsilon_1 \cos\varphi . R_{12} + \sin\varphi . R_{22} ,$$
$$\varepsilon_1 \cos\varphi . R^*_{21} - \varepsilon_1 \sin\varphi . R^*_{22} = - \varepsilon_1 \sin\varphi . R_{11} + \cos\varphi . R_{21} ,$$
$$\sin\varphi . R^*_{21} + \cos\varphi . R^*_{22} = - \varepsilon_1 \sin\varphi . R_{12} + \cos\varphi . R_{22} ,$$

i.e.,

$$(17) \qquad R^*_{11} = \cos^2\varphi . R_{11} + \varepsilon_1 \sin\varphi \cos\varphi . R_{12} + \varepsilon_1 \sin\varphi \cos\varphi . R_{21} + \sin^2\varphi . R_{22} ,$$
$$R^*_{12} = - \sin\varphi \cos\varphi . R_{11} + \varepsilon_1 \cos^2\varphi . R_{12} - \varepsilon_1 \sin^2\varphi . R_{21} + \sin\varphi \cos\varphi . R_{22} ,$$
$$R^*_{21} = - \sin\varphi \cos\varphi . R_{11} - \varepsilon_1 \sin^2\varphi . R_{12} + \varepsilon_1 \cos^2\varphi . R_{21} + \sin\varphi \cos\varphi . R_{22} ,$$

$$R^*_{22} = \sin^2\varphi \,.R_{11} - \varepsilon_1 \sin\varphi \cos\varphi \,.R_{12} -$$
$$- \varepsilon_1 \sin\varphi \cos\varphi \,.R_{21} + \cos^2\varphi \,.R_{22} \ .$$

Introduce the form

$$(18) \qquad \Phi = (R_1R_{21} - R_2R_{11})\omega^1 + (R_1R_{22} - R_2R_{12})\omega^2 \ .$$

From (14) and (17),

$$R^*_1R^*_{21} - R^*_2R^*_{11} = \cos\varphi \,.(R_1R_{21} - R_2R_{11}) +$$
$$+ \varepsilon_1 \sin\varphi \,.(R_1R_{22} - R_2R_{12}) \ ,$$
$$R^*_1R^*_{22} - R^*_2R^*_{12} = - \sin\varphi \,.(R_1R_{21} - R_2R_{11}) +$$
$$+ \varepsilon_1 \cos\varphi \,.(R_1R_{22} - R_2R_{12}) \ ,$$

i.e.,

$$(19) \qquad \Phi^* = \varepsilon_1 \Phi \quad ,$$

and Φ is an invariant 1-form (associated to φ) on oriented surfaces.

From (15),

$$(20) \quad \{dR_{11} - (R_{12} + R_{21})\omega_1^2\} \wedge \omega^1 + \{dR_{12} +$$
$$+ (R_{11} - R_{22})\omega_1^2\} \wedge \omega^2 = R_2K\,\omega^1 \wedge \omega^2 \quad ,$$

$$\{dR_{21} + (R_{11} - R_{22})\omega_1^2\} \wedge \omega^1 + \{dR_{22} +$$
$$+ (R_{12} + R_{21})\omega_1^2\} \wedge \omega^2 = - R_1K\,\omega^1 \wedge \omega^2$$

and we get the existence of functions $S_1, \dots, S_6$ such that

$$(21) \quad dR_{11} - (R_{12} + R_{21})\omega_1^2 = S_1\omega^1 + (S_2 - \tfrac{1}{2} KR_2)\omega^2 \quad ,$$

$$dR_{12} + (R_{11} - R_{22})\omega_1^2 = (S_2 + \frac{1}{2} KR_2)\omega^1 + S_3\omega^2 ,$$

$$dR_{21} + (R_{11} - R_{22})\omega_1^2 = S_4\omega^1 + (S_5 + \frac{1}{2} KR_1)\omega^2 ,$$

$$dR_{22} + (R_{12} + R_{21})\omega_1^2 = (S_5 - \frac{1}{2} KR_1)\omega^1 + S_6\omega^2 .$$

From this

(22) $$d\Phi = (2 R_{11}R_{22} - 2 R_{12}R_{21} - R_1^2K - R_2^2K)\omega^1\wedge\omega^2 ,$$

and we get the desired integral formula

(23) $$\int_{\partial M} \{(R_1R_{21} - R_2R_{11})\omega^1 + (R_1R_{22} - R_2R_{12})\omega^2\} =$$

$$= \int_M \{2(R_{11}R_{22} - R_{12}R_{21}) - (R_1^2 + R_2^2)K\}\omega^1\wedge\omega^2 .$$

3. Let us return to our surface M. Because of $K > 0$ let us choose an orientation of the normal and of M itself, i.e.,

(24) $$\varepsilon_1 = \varepsilon_2 = 1 .$$

Consider the 1-form

(25) $$\psi = (\alpha+\gamma)\omega^1 + (\beta+\delta)\omega^2 = 2\, dH$$

which is, according to (10) and (12) resp., invariant on M. From (6),

(26) $$d(\alpha+\gamma) - (\beta+\delta)\omega_1^2 = (A + C + cK)\omega^1 + (B + D)\omega^2 ,$$

$$d(\beta+\delta) + (\alpha+\gamma)\omega_1^2 = (B + D)\omega^1 + (C + E + aK)\omega^2 .$$

Applying the integral formula (23), we get

$$(27)\quad \int_{\partial M}\{(\alpha+\gamma)\,(B + D) - (\beta+\delta)(A + C + cK)\}\,\omega^1 +$$

$$+ \int_{\partial M}\{(\alpha+\gamma)\,(C + E + aK) - (\beta+\delta)\,(B + D)\}\,\omega^2 =$$

$$= \int_M \{ 2(A + C + cK)\,(C + E + aK) - 2(B + D)^2 -$$

$$- (\alpha+\gamma)^2 K - (\beta+\delta)^2 K \}\,\omega^1\wedge\omega^2 \ .$$

The frames be chosen in such a way that $V_1 = v_1$, $V_2 = v_2$. From (25),

$$(28)\qquad V_1H = \frac{1}{2}\,(\alpha+\gamma)\ ,\qquad V_2H = \frac{1}{2}\,(\beta+\delta)\ ,$$

i.e.,

$$(29)\qquad \beta + \delta = 0 \quad \text{on}\quad M$$

and

$$(30)\qquad \alpha + \gamma = \beta + \delta = 0 \quad \text{on}\quad \partial M \ .$$

From (26) and (29),

$$(31)\qquad V_1V_1H = \frac{1}{2}\,V_1(\alpha+\gamma) = \frac{1}{2}\,(A + C + cK) = 0 \quad \text{on}\quad M \ .$$

Thus the integral formula (27) reduces to

$$(28)\qquad 0 = \int_M \{ 2(B + D)^2 + (\alpha+\gamma)^2 K \}\,\omega^1\wedge\omega^2\ ,$$

and we get

$$(29)\qquad V_1H = \frac{1}{2}\,(\alpha+\gamma) = 0 \quad \text{on}\quad M\ ,$$

i.e., H = const. on M . QED.

__Remark.__ Notice that our Theorem is non-trivial. Indeed, let us show that __there are, locally, surfaces of class__ C^∞ __possessing two orthogonal unit tangent vector fields__ V_1, V_2 __such that__ (1) __is valid and__ H __is not a constant on__ M.

The prolongation of (7) yields

$$
\begin{aligned}
(30)\quad & \Delta A \wedge \omega^1 + \Delta B \wedge \omega^2 = (4\beta K + bK_1)\,\omega^1 \wedge \omega^2 , \\
& \Delta B \wedge \omega^1 + \Delta C \wedge \omega^2 = (3\gamma K - 2\alpha K - aK_1 + \\
& \qquad + bK_2)\,\omega^1 \wedge \omega^2 , \\
& \Delta C \wedge \omega^1 + \Delta D \wedge \omega^2 = (2\delta K - 3\beta K - \\
& \qquad - bK_1 + cK_2)\,\omega^1 \wedge \omega^2 , \\
& \Delta D \wedge \omega^1 + \Delta E \wedge \omega^2 = -(4\gamma K + bK_2)\,\omega^1 \wedge \omega^2
\end{aligned}
$$

with $dK = K_1\omega^1 + K_2\omega^2$ and

$$
\begin{aligned}
(31)\quad & \Delta A := dA - 2(2B + bK)\,\omega_1^2 = F_1\omega^1 + F_2\omega^2 , \\
& \Delta B := dB + (A - 3C - 2aK - cK)\,\omega_1^2 = \\
& \qquad = (F_2 + 4\beta K + bK_1)\,\omega^1 + F_3\omega^2 , \\
& \Delta C := dC + 2(B - D)\,\omega_1^2 = (F_3 + 3\gamma K - 2\alpha K - \\
& \qquad - aK_1 + bK_2)\,\omega^1 + F_4\omega^2 , \\
& \Delta D := dD + (3C - E + aK + 2cK)\,\omega_1^2 = \\
& \qquad = (F_4 + 2\delta K - 3\beta K - bK_1 + cK_2)\,\omega^1 + F_5\omega^2 , \\
& \Delta E := dE + 2(2D + bK)\,\omega_1^2 = (F_5 - 4\gamma K - \\
& \qquad - bK_2)\,\omega^1 + F_6\omega^2 ,
\end{aligned}
$$

$F_1, \dots, F_6$ being new functions. Our surfaces are then given by the system (6),(9),(31). The system (6) reduces to

$$(32) \qquad d\alpha - 3\beta\omega_1^2 = A\omega^1 + (B - bK)\omega^2 ,$$

$$d\beta + (\alpha - 2\gamma)\omega_1^2 = (B + bK)\omega^1 + (aK - cK - A)\omega^2 ,$$

$$d\gamma + 3\beta\omega_1^2 = - A\omega^1 + (D + bK)\omega^2 ,$$

$$(\alpha+\gamma)\omega_1^2 = (B + D)\omega^1 + (aK - cK - A + E)\omega^2 ,$$

$\alpha + \gamma \neq 0$ because of $H \neq$ const.

From (31),

$$(33) \qquad \Delta C = - \Delta A - 2(B + D)\omega_1^2 - (\gamma K + cK)\omega^1 +$$
$$+ (\beta K - cK_2)\omega^2 =$$
$$= - \Delta A - 2(B + D)\ (\alpha+\gamma)^{-1}\{(B + D)\omega^1 +$$
$$+ (aK - cK - A + E)\omega^2\} - (\gamma K + cK_1)\omega^1 +$$
$$+ (\beta K - cK_2)\omega^2 ,$$

and the differential consequences of (32) are

$$(34) \qquad \Delta A \wedge \omega^1 + \Delta B \wedge \omega^2 = (4\beta K + bK_1)\omega^1 \wedge \omega^2 \equiv$$
$$\equiv f_1 \omega^1 \wedge \omega^2 ,$$

$$\Delta B \wedge \omega^1 - \Delta A \wedge \omega^2 = \{2(2\gamma - \alpha)K + (c - a)K_1 + bK_2 +$$
$$+ 2(\alpha+\gamma)^{-1}(B + D)^2\}\omega^1 \wedge \omega^2 \equiv f_2\omega^1 \wedge \omega^2 ,$$

$$- \Delta A \wedge \omega^1 + \Delta D \wedge \omega^2 = -\{4\beta K + bK_1 +$$
$$+ 2(\alpha+\gamma)^{-1}(B + D)(aK - cK - A + E)\}\omega^1 \wedge \omega^2 \equiv f_3\omega^1 \wedge \omega^2 ,$$

$$\Delta D \wedge \omega^1 + \Delta E \wedge \omega^2 = - (4\gamma K + bK_2)\omega^1 \wedge \omega^2 \equiv f_4\omega^1 \wedge \omega^2$$

with

$$(35)\quad \Delta A = G_1\omega^1 + G_2\omega^2 , \quad \Delta B = (G_2 + f_1)\omega^1 - (G_1 + f_2)\omega^2 ,$$

$$\Delta D = (f_3 - G_2)\omega^1 + G_3\omega^2 , \quad \Delta E = (G_3 + f_4)\omega^1 + G_4\omega^2 .$$

The system is in involution and the considered surfaces depend (in the usual sense of E. Cartan) on 4 functions o 1 variable.

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