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MINIMAL REALIZATIONS FOR FINITE SETS IN CATEGORIAL AUTOMATA THEORY

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Abstract: The minimal realization problem in the category Set is solved for finite or bounded sets.

Key words: Set functor, free algebra, minimal realization.

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Let a cardinal $m > 1$ be given. In the present note, we characterize all functors X: Set $\longrightarrow$ Set such that for each set I with card $I < m$ there exists a free X-algebra over I and each mapping $f: X^{\circledast} I \longrightarrow 2$ has a minimal realization (see 4 and 6). Some simple criteria and examples are given (see 7, 8 and 9).

The present note is related to the paper [7], where all input processes in Set are characterized and to [9], where the minimal realization problem is solved in Set without respect to cardinalities. The minimal realization problem in more general categories is investigated in [1], [2],[5].

1. Preliminaries: a) We use the term set functor for a covariant functor X: Set $\longrightarrow$ Set, Set being the

category of all sets, such that $XQ \neq \emptyset$ whenever $Q \neq \emptyset$. If $Q \subset Q'$ are sets, denote by $i: Q \longrightarrow Q'$ the inclusion map $i(x) = x$ for $x \in Q$. X is said to preserve inclusions if $\emptyset \neq Q \subset Q'$ implies $XQ \subset XQ'$ and Xi is inclusion again. By [3], each set functor is naturally equivalent to an inclusion preserving functor. Since we always work with functors "up to natural equivalence" we shall assume, in what follows, that all functors considered preserve inclusions.

b) Let X be a set functor, $\mathfrak{m}$ be a positive cardinal. Define functors $X_{[<\mathfrak{m}]}$, $X_{[\leq \mathfrak{m}]}$ by

$$X_{[<\mathfrak{m}]}Q = \bigcup_{\substack{\emptyset \neq P \subset Q \\ \text{card } P < \mathfrak{m}}} XP \text{ whenever } Q \neq \emptyset, \quad X_{[<\mathfrak{m}]}\emptyset = X\emptyset,$$

$$X_{[\leq\mathfrak{m}]}Q = \bigcup_{\substack{\emptyset \neq P \subset Q \\ \text{card } P \leq \mathfrak{m}}} XP \text{ whenever } Q \neq \emptyset, \quad X_{[\leq\mathfrak{m}]}\emptyset = X\emptyset,$$

$X_{[<\mathfrak{m}]}f$ and $X_{[\leq\mathfrak{m}]}f$ are the domain-range-restrictions of Xf for any mapping f. If $\text{card } Q = \mathfrak{m}$, denote

$$Q_X = XQ \setminus X_{[<\mathfrak{m}]}Q .$$

If $Q_X \neq \emptyset$, then $\mathfrak{m}$ is called an unattainable cardinal of X (see [6]). We recall that a set functor X is finitary iff $X = X_{[<\aleph_0]}$, i.e. iff it has no infinite unattainable cardinals. It is well known that finitary functors may be characterized as factor-functors of $\coprod_{j \in \mathcal{J}} \text{Hom}(Q_j, -)$, where $\mathcal{J}$ is a non-empty set and all Q_j are finite sets. The characterization of finitary functors by means of the preservation of colimits or limits of certain diagrams is given in [2],[5],[9].

c) Let X be a set functor. The category $\mathcal{D}yn\ X$ (see [4]) is defined as follows. Objects (called X-dynamics) are all pairs (Q, σ), such that Q is a set, $\sigma: XQ \longrightarrow Q$ is a mapping; morphisms (called X-dynamorphisms) $f: (Q, \sigma) \longrightarrow (Q', \sigma')$ are mappings from Q in Q' such that $f \circ \sigma = \sigma' \circ (Xf)$. A free X-algebra over Q is a triple $X_A Q = (X^@ Q, \omega_Q, \eta_Q)$, where $(X^@ Q, \omega_Q)$ is an X-dynamics and $\eta_Q : Q \longrightarrow X^@ Q$ is a mapping with the universal property, i.e. for each X-dynamics (Q', σ') and each mapping $g: Q \longrightarrow Q'$ there exists exactly one X-dynamorphism $\overline{g} : (X^@ Q, \omega_Q) \longrightarrow (Q', \sigma')$ such that $g = \overline{g} \circ \eta_Q$. A construction of $X_A Q$ is given in [7] as follows.

$$X_0 Q = Q \times \{0\},$$

$$X_1 Q = X_0 Q \cup (XX_0 Q \times \{1\})$$

whenever $Q \neq \emptyset$, $X_1 \emptyset = ((X\vartheta) X\emptyset) \times \{1\}$, where $\vartheta : \emptyset \longrightarrow 1$ is the empty mapping,

$$X_\alpha Q = \bigcup_{\beta < \alpha} X_\beta Q \quad \text{whenever } \alpha \text{ is a limit ordinal,}$$

$$X_{\alpha+1} Q = X_\alpha Q \cup (XX_\alpha Q \setminus \bigcup_{\beta < \alpha} XX_\beta Q) \times \{\alpha + 1\}.$$

By [7], $X_A Q$ exists iff this process stops, i.e. $X_{\alpha+1} Q = X_\alpha Q$ for some α. Then $X^@ Q = X_\alpha Q$ and $\eta_Q : Q \longrightarrow X^@ Q$ is defined by $\eta_Q(q) = (q,0)$, $\omega_Q : XX^@ Q \longrightarrow X^@ Q$ by $\omega_Q(q) = (q, \beta + 1)$, where β is the smallest ordinal such that $q \in XX_\beta Q$.

d) Let X be a set functor, I be a set such that $X_A I$ does exist. Let $f: X^{@} I \longrightarrow Y$ be a mapping. The following notions and their interpretation in the automata theory are given in [4]. An X-dynamorphism $r: (X^{@} I, \omega_I) \longrightarrow (Q, \delta)$ is called *a reachable realization of* f *in* $\mathcal{D}yn\, X$ if it is a mapping onto Q and f factorizes through r . It is called a *minimal realization of* f *in* $\mathcal{D}yn\, X$ if it is a reachable realization of f which factorizes through any reachable realization of f .

2. *Proposition*: Let X be a set functor, I be a set such that $X_A I$ does exist. Let there be no infinite unattainable cardinal of X smaller than or equal to card $X^{@} I$. Then each mapping $f: X^{@} I \longrightarrow Y$, Y is a set, has a minimal realization.

Proof. Put ψ = card $X^{@} I$ and replace X by $X_{[\leq \psi]}$. Since $X_{[\leq \psi]}$ is finitary, each mapping $f:$ $: X^{@} I \longrightarrow Y$ has a minimal realization in $\mathcal{D}yn\, X_{[\leq \psi]}$, by [9] , so in $\mathcal{D}yn\, X$.

3. *Proposition*: Let X be a set functor, $\aleph$ be its infinite unattainable cardinal. Let I be a set such that $X_A I$ does exist and card $X^{@} I \geq \aleph$. Then there exists a mapping $f: X^{@} I \longrightarrow 2$ which has no minimal realization in $\mathcal{D}yn\, X$.

Proof. a) By [7], $X^{@} I = \bigcup_{\alpha < \lambda} X_\alpha I$, where λ is an ordinal. There exists $\alpha < \lambda$ such that card $X_\alpha I \geq$ $\geq \varkappa$ (otherwise the process could not stop at λ because card $X\varkappa > \varkappa$, see [6]). Let γ be the smallest ordinal such that card $X_\gamma I \geq \varkappa$. Then

$$card\,(X_{\gamma+1} I \setminus X_\gamma I) = card\,(X X_\gamma I \setminus \bigcup_{\beta<\gamma} X X_\beta I) > \varkappa,$$

see [6]. Choose a set P with card $P = \varkappa$ and a point a not in P and such that the sets $X_\gamma I$, $P \cup \{a\}$, $(P \cup \{a\}) \times \{0,1\}$ are disjoint. Put

$$Q = X_\gamma I \cup (P \cup \{a\}) \times \{0,1\} .$$

b) Denote by $\mathbb{F}$ the set of all non-empty finite subsets of P. If $F \in \mathbb{F}$ put $Q_F = X_\gamma I \cup F \cup ((P \setminus F) \cup$ $\cup \{a\}) \times \{0,1\}$, $g_F: Q \longrightarrow Q_F$ is the mapping given by $g_F((p,i)) = p$ whenever $p \in F$, $i = 0,1$, $g_F(q) = q$ otherwise. If $F \subset F' \in \mathbb{F}$ denote by $g^F_{F'}: Q_F \longrightarrow Q_{F'}$ the mapping such that $g_{F'} = g^F_{F'} \circ g_F$. We recall that P_X is defined as $P_X = XP \setminus \bigcup_{\substack{\emptyset \neq R \subset P \\ card\, R < card\, P}} XR$. Since card $P = \varkappa$ and this is an unattainable cardinal of X, we have $P_X \neq \emptyset$. Let $v_0, v_1: P \longrightarrow Q$ be the mappings given by $v_i(p) =$ $= (p,i)$. Put $A^i = (X v_i) P_X$, $A^i_F = (X(g_F \circ v_i)) P_X$. Thus, if $F \subset F' \in \mathbb{F}$, then $A^i_{F'} = (X g^F_{F'}) A^i_F$. Since $g_F(v_0(P)) \cap$ $\cap g_F(v_1(P))$ is finite, we have $A^0_F \cap A^1_F = \emptyset$. Put

$$B^i = \bigcup_{F \in \mathbb{F}} (X g_F)^{-1} A^i_F , \quad B^i_F = \bigcup_{\substack{F' \in \mathbb{F} \\ F' \supset F}} (X g^F_{F'})^{-1} A^i_{F'} .$$

Then $B^0 \cap B^1 = \emptyset$, $B_F^0 \cap B_F^1 = \emptyset$. Since X preserves inclusions, we have $XX_\gamma I \subset XQ$ and $XX_\gamma I \subset XQ_F$. Since $X_\gamma I \cap g_F(v_0(P))$ is finite (it is empty!), we have $XX_\gamma I \cap \cap A_F^i = \emptyset$ for all $F \in \mathbf{F}$, so $XX_\gamma I \cap B^i = \emptyset$ for $i = 0,1$. We define

$$\sigma : XQ \longrightarrow Q$$

as follows:

$$\sigma(z) = \omega_I(z) \quad \text{whenever} \quad z \in \bigcup_{\beta<\gamma} XX_\beta I ,$$

σ maps $XX_\gamma I \setminus \bigcup_{\beta<\gamma} XX_\beta I$ onto Q (arbitrarily),

$$\sigma(z) = (a,1) \quad \text{whenever} \quad z \in B^1 ,$$

$$\sigma(z) = (a,0) \quad \text{whenever} \quad z \in XQ \setminus (XX_\gamma I \cup B^1) .$$

Thus, (Q,σ) is an X-dynamics.

c) We have $X_0 I = I \times \{0\} \subset X_\gamma I \subset Q$. Define a mapping $g_0: I \longrightarrow Q$ by $g_0(x) = (x,0)$. Let $g: (X^{@} I, \omega_I) \longrightarrow (Q,\sigma)$ be the X-dynamorphism such that $g \circ \eta_I = g_0$. We show that g is a mapping onto Q . Since $g(y) = y$ for all $y \in X_0 I$ and $\sigma(z) = \omega_I(z)$ whenever $z \in \bigcup_{\beta<\gamma} XX_\beta I$, we have $g(y) = y$ for all $y \in X_\gamma I$. Since σ maps $XX_\gamma I$ onto Q , g maps $XX_{\gamma+1} I$ onto Q , so it maps $X^{@} I$ onto Q .

d) Let $h: Q \longrightarrow 2$ be the mapping such that $h((a,1)) = 1$, $h(q) = 0$ otherwise. Define $f: X^{@} I \longrightarrow 2$ by $f = h \circ g$. We show that f has no minimal realization in $\mathcal{Dyn}\, X$. By c), g is a reachable realization of f . First, we show that, for each $F \in \mathbb{F}$, $g_F \circ g$ is a reachable realization of f . Clearly, h factorizes through g_F , so f factorizes through $g_F \circ g$ and this is a mapping onto Q_F . Hence, it is sufficient to find $\sigma_F :$ $: XQ_F \longrightarrow Q_F$ such that $g_F \circ \sigma = \sigma_F \circ (X g_F)$. Put $\sigma_F(z) =$ $= g_F \circ \sigma(z)$ whenever $z \in XX_{\gamma} I$, $\sigma_F(z) = (a,1)$ whenever $z \in B_F^1$, $\sigma_F(z) = (\alpha, 0)$ otherwise.

e) Let us suppose that f has a minimal realization in $\mathcal{Dyn}\, X$, say $t: (X^{@} I, \omega_I) \longrightarrow (R, \varphi)$. Since t factorizes through each $g_F \circ g$, $F \in \mathbb{F}$, it factorizes through the mapping $k \circ g$, where $k: Q \longrightarrow X_{\gamma} I \cup P \cup$ $\cup \{(\alpha,0), (\alpha,1)\}$ is defined by $k((p,i)) = p$ whenever $p \in P$, $i = 0,1$, $k(z) = z$ otherwise. Choose $q \in P_X$ and put $q_i = (X v_i)(q) \in A^i$. Find $p_i \in XX^{@} I$ so that $q_i =$ $= (Xg)(p_i)$. We have $(X(k \circ g))(p_0) = (Xk)(q_0) =$ $= (X(k \circ v_0))(q) = (X(k \circ v_1))(q) = (Xk)(q_1) =$ $= (X(k \circ g))(p_1)$, so $(\varphi \circ Xt)(p_0) = (\varphi \circ Xt)(p_1)$. On the other hand, $\sigma(q_0) = (\alpha, 0)$, $\sigma q_1 = (\alpha, 1)$ and $h((\alpha,0)) \neq h((\alpha,1))$, so $(t \circ \omega_I)(p_0)$ and $(t \circ \omega_I)(p_1)$ must be distinct, which is a contradiction.

4. Denote by N the set of all positive integers.

<u>Theorem</u>. Let X be a set functor, m, n be cardinals such that $0 \leq m < n \geq \aleph_0$. The following statements are equivalent.

($1_{\mathfrak{m}}$) For each set I with card $I < \mathfrak{n}$ there exists $X_A I$ and each mapping $f: X^{@} I \longrightarrow Y$, Y arbitrarily, has a minimal realization in $\mathcal{D}yn\ X$.

($2_{\mathfrak{m}}$) For each set I with $\mathfrak{m} \leq$ card $I < \mathfrak{n}$ there exists $X_A I$ and each mapping $f: X^{@} I \longrightarrow 2$ has a minimal realization in $\mathcal{D}yn\ X$.

($3_{\mathfrak{m}}$) $X_{[<\mathfrak{p}]}$ is finitary, where $\mathfrak{p} = \max(\mathfrak{n}, \mathfrak{q})$, $\mathfrak{q}'$ is the smallest cardinal greater than $\sup_{n \in N}$ card Xn.

<u>Proof</u>. Clearly, ($1_{\mathfrak{m}}$) $\Longrightarrow$ ($2_{\mathfrak{m}}$). The implication ($3_{\mathfrak{m}}$) $\Longrightarrow$ ($1_{\mathfrak{m}}$) follows from 2 and Lemma A below, the implication non ($3_{\mathfrak{m}}$) $\Longrightarrow$ non($2_{\mathfrak{m}}$) follows from 2 and Lemma B below.

5. <u>Lemma A</u>: Let $X_{[<\mathfrak{p}]}$ be finitary. If I is a set with $0 <$ card $I < \mathfrak{n}$, then $X_A I$ exists and card $X^{@} I < \mathfrak{p}$.

<u>Proof</u>. Put $r = (\text{card } I \times \sup_{n \in N} \text{card } Xn) + \aleph_0$.

Since $X_{[<\mathfrak{p}]}$ is finitary, $XI = \bigcup_{\substack{\emptyset \neq F \subset I \\ F \text{ finite}}} XF$, so card $XI \leq r \leq \mathfrak{p}$. Suppose $\mathfrak{p} > \aleph_0$. (The case $\mathfrak{p} = \aleph_0$ is easy, see [6].) Since $X_{[<\mathfrak{p}]}$ is finitary, card $XQ \leq r$ whenever $0 <$ card $Q \leq r$. We may prove by induction that card $X_n I \leq r$ for all $n \in N$, so card $X_{\omega_0} I \leq r < \mathfrak{p}$. Since $X_{[<\mathfrak{p}]}$ is finitary, the process stops at $X_{\omega_0} I$.

Lemma B: Let r be an unattainable cardinal of X such that $\aleph_0 \leq r < \mathfrak{g}$. Then there exists a set I with $\mathfrak{m} \leq \text{card}\, I < \mathfrak{n}$ and $\text{card}\, X_{\omega_0} I \geq r$.

Proof. a) Let $\mathfrak{g} = \mathfrak{n} \geq \mathfrak{g}$. Then $\aleph_0 \leq r < \mathfrak{n}$. Choose I with $\text{card}\, I = \max(r, \mathfrak{m})$ and use $I \times \{0\} = X_0 I \subset$ $\subset X_{\omega_0} I$.

b) Let $\mathfrak{g} = \mathfrak{g}' > \mathfrak{n}$. Put $s = \sup_{n \in N} \text{card}\, Xn$. Then $\aleph_0 \leq r \leq s$.

α) If $\aleph_0 \leq r < s$, then there exists $n \in N$ such that $\text{card}\, Xn \geq r$. Put $I = \max(n, \mathfrak{m})$. Then $\text{card}\, X_{\omega_0} I \geq r$.

β) If $\aleph_0 < r = s$, then there exists $n \in N$ such that $\text{card}\, Xn \geq \aleph_0$.

Then $\text{card}\, X_2 n \geq \text{card}\, XXn \geq r$. Put $I = \max(n, \mathfrak{m})$.

γ) Let $\aleph_0 = r = s$. If $n > \text{card}\, Xn$ for some $n \in N$, then, by [6], X is constant on $\{1, 2, \ldots, n-1\}$. Since $\sup_{n \in N} \text{card}\, Xn = \aleph_0$, there exists $k \in N$ such that $\text{card}\, Xn \geq n$ for all $n = k, k+1, \ldots$. Choose $I =$ $= \max(k, \mathfrak{m})$. Then $\text{card}\, X_0 I = I$ and, by induction, $\text{card}\, X_n I \geq \text{card}\, XX_{n-1} I \geq I + n$, so $\text{card}\, X_{\omega_0} I \geq \aleph_0 = r$.

6. Denote by C_M a constant set functor, i. e. $C_M Q = M$ for each set Q, $C_M f$ is the identity for any

mapping f . If X is a set functor, put

$$X^1 = X \ , \quad X^{n+1} = XX^n \ .$$

If X, X' are set functors, then their coproduct is denoted by $X \vee X'$.

<u>Theorem</u>. Let X be a set functor. Let an n in N be given. The following assertions are equivalent.

(4_n) For each set I with $\operatorname{card} I \leq n$ there exists $X_A I$ and each mapping $f: X^{@} I \longrightarrow Y$, Y arbitrarily, has a minimal realization in $\mathcal{D}yn\ X$.

(5_n) There exists $X_A n$ and each mapping $f: X^{@} n \longrightarrow 2$ has a minimal realization in $\mathcal{D}yn\ X$.

(6_n) $X_{[\leq q]}$ is finitary, where
$q = \sup_{k \in N} \operatorname{card} (C_n \vee X)^k n$.

<u>Proof</u>. $(6_n) \Longrightarrow (4_n)$: Clearly, $\operatorname{card} X_k I = \operatorname{card} (C_n \vee X)^k I$ for all $k \in N$ whenever $\operatorname{card} I = n$. Hence, $\operatorname{card} X_{\omega_0} I \leq q$ whenever $\operatorname{card} I \leq n$. Since $X_{[\leq q]}$ is finitary, $X_A I$ there exists and $X^{@} I = X_{\omega_0} I$. Then use 2.

$(4_n) \Longrightarrow (5_n)$ is evident.

$\operatorname{non}(6_n) \Longrightarrow \operatorname{non}(5_n)$: Since $X_{[\leq q]}$ is not finitary, there exists an unattainable cardinal r of X such that $\aleph_0 \leq r \leq q$. We have $\operatorname{card} X_k n = \operatorname{card} (C_n \vee X)^k n$, so $\operatorname{card} X_{\omega_0} n \geq r$, hence $\operatorname{card} X^{@} n$, whenever $X_A n$ exists, is greater than or equal to r . Now, use 3.

7. Theorem. Let X be a set functor such that

$$\operatorname{card} X\,1 < \operatorname{card} X\,2 .$$

Then all the statements $(1_{\aleph_0})$ - $(3_{\aleph_0})$, (4_1) - (6_1) are equivalent.

Proof. If $\aleph_0 > \operatorname{card} X\,2 > \operatorname{card} X\,1$, then, by [6], $\operatorname{card} X(n+1) > \operatorname{card} X\,n$ for all $n \in N$. Hence, $s = \sup_{n \in N} \operatorname{card} X\,n \geq \aleph_0$, so $X_{[<\aleph_0]} = X_{[\leq s]}$.

$(3_{\aleph_0}) \Longrightarrow (4_1)$: Let us suppose that $X_{[\leq s]}$ is finitary. Then use 5A and 2 for $I = 1$.

$(6_1) \Longrightarrow (3_{\aleph_0})$: We recall that

$q = \sup_{n \in N} \operatorname{card} (C_1 \vee X)^n\,1$.

It is sufficient to prove that $q \geq s$.

a) First, we show that $\operatorname{card} X_n 1 \geq \operatorname{card} XX_{n-1} 1$ for all $n \in N$. Clearly, $\operatorname{card} X_1 1 \geq \operatorname{card} XX_0 1$. Now, by the induction hypothesis, $\operatorname{card} X_n 1 \geq \operatorname{card} XX_{n-1} 1$, hence $\operatorname{card} X_{n+1} 1 \geq \operatorname{card} XX_n 1$.

b) Now, we show that $X_n 1 \subsetneq X_{n+1} 1$ and $XX_n 1 \subsetneq XX_{n+1} 1$. It is easy to prove it by induction because the following statement is fulfilled (see [6]): if $\operatorname{card} X\,2 > \operatorname{card} X\,1$, then $XQ \subsetneq XQ'$ whenever $\emptyset \neq Q \subsetneq Q'$.

c) Now, we prove by induction that $\operatorname{card} X_n 1 \geq n$. Clearly, $\operatorname{card} X_1 1 \geq 1$; by b) and the inclusion hypothesis $XX_n 1 \setminus XX_{n-1} 1 \neq \emptyset$ and $\operatorname{card} X_n 1 \geq n$, so $\operatorname{card} X_{n+1} 1 \geq$ $\geq n + 1$.

d) Finally, we show card $X_{n+1}1 \geq$ card Xn . Since card $X_n\,1 \geq$ card $XX_{n-1}\,1$, we have card $X_{n+1}\,1 \geq$ card $XX_n 1$. Since card $X_n\,1 \geq n$, card $X_{n+1}\,1 \geq$ card Xn .

8. Now, we give an easy, but very simple condition for the validity of $(1_{\aleph_0})$ - (6_1) for some special set functors.

Proposition. Let X be a set functor such that card $X\,2 >$ card $X\,1$ and, for any $n \in N$, Xn is finite. Then all the statements $(1_{\aleph_0})$ - $(3_{\aleph_0})$, (4_1) - (6_1) are equivalent to

(7) $$\text{card } X\aleph_0 \leq \aleph_0 .$$

Proof. If card $X\,2 >$ card $X\,1$ and all Xn are finite, then $s = \sup_{n \in N} \text{card } Xn = \aleph_0$. $X_{[\leq s]}$ is finitary iff s is not an unattainable cardinal of X . If s is an unattainable cardinal of X , then card $X\,s > \aleph_0$, by [6]. If it is not, then $X\,s = \bigcup_{n \in N} X\,n = \aleph_0$. Thus, $(3_{\aleph_0})$ is equivalent to (7).

9 . Remark. In a draft of his book Algeoraic Theories, E.G. Manes put the question whether there exists an input process X ($= X_A I$ does exist for any I) in Set such that for some finite sets I, Y there exists $f: X^{@} I \longrightarrow Y$ which has no minimal realization in $\mathcal{Dyn}$ X . By 8, the functors $X = \beta_{[\leq \aleph_0]}$ or $X = \mathbb{N}_{[\leq \aleph_0]}$ (the functors β , $\mathbb{N}$ are described, for example, in [8]) are

such input processes (small functors are input processes, see [7]). Another example is the functor $X = \mathrm{Hom}(\aleph_0, -)$. Here, card $X\,1 = 1$, card $X\,2 = 2^{\aleph_0}$ and $\sup_{n \in N}$ card $X\,n = 2^{\aleph_0}$, while $\aleph_0$ is an unattainable cardinal of X . By 7, there exists a mapping $f: X^{\circledast} 1 \longrightarrow 2$ which has no minimal realization in $\mathcal{D}yn\ X$.

By 4 and 6, one can prove easily the following assertion: for any cardinal $m > 1$ there exists an input process X such that, for each set I , card $I < m$ iff each mapping $f: X^{\circledast} I \longrightarrow 2$ has a minimal realization in $\mathcal{D}yn\ X$.

10. In the present note, we restrict ourselves to the category Set only. But analogous results are true, for example, in the category Vect of all vector spaces (over a field R) and all linear mappings. Here, we consider additive endo-functors X: Vect $\longrightarrow$ Vect only. The presented theorems remain true for vector spaces if we write dim instead of card (also in the definition of $X_{[<m]}$) and R instead of 2 . The proofs may be modified such that we take, roughly speaking, suitable bases of vector spaces considered or, conversely, required vector spaces are defined as linear envelopes of suitable sets and, analogously, for mappings. (Certainly, other easy modifications are also necessary, for example the set $2 = \{0,1\}$ is considered as a subset of the field R , constant functors are not additive, but they may be used for the defi-

nition of q and so on.)

References:

[1] J. ADÁMEK: Free algebras and automata realization in the language of categories, Comment.Math.Univ. Carolinae 15(1974),589-602.

[2] J. ADÁMEK: Realization theory for automata in categories, to appear.

[3] J. ADÁMEK, V. KOUBEK, V. POHLOVÁ: The colimits in the generalized algebraic categories, Acta Univ.Carolinae 13(1972),29-40.

[4] A. ARBIB, E.G. MANES: A categorist's view of automata and systems, Category Theory applied to Computation and Control,Proceedings of the First International Symposium,Amherst,Massachussetts 1974,62-78.

[5] M. BARR: Right exact functors, J.of Pure and Applied Algebra 4(1974),1-

[6] V. KOUBEK: Set functors, Comment.Math.Univ.Carolinae 12(1971),175-195.

[7] V. KURKOVÁ-POHLOVÁ, V. KOUBEK: When a generalized algebraic category is monadic, Comment.Math.Univ. Carolinae 15(1974),577-587,

[8] V. TRNKOVÁ: On descriptive classification of set functors I, Comment.Math.Univ.Carolinae 12(1971), 143-174.

[9] V. TRNKOVÁ: On minimal realizations of behavior maps in categorial automata theory, Comment.Math. Univ.Carolinae 15(1974),555-566.

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