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SOME REMARKS ON NON-SEPARABLE BANACH SPACES WITH MARKUŠEVIČ BASIS

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Abstract: If a Banach space X has a Markuševič basis $\{x_i\}$ whose coefficient space is norming, then X has an equivalent locally uniformly rotund norm and $\{x_i\}$ contains a basic subset of the same cardinality. Certain Banach spaces are observed to be Lindelöf in its weak topology.

Key words: Markuševič basis, rotundity, Lindelöf space.

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J. Lindenstrauss and D. Amir and J. Lindenstrauss have constructed in weakly compactly generated (WCG) Banach spaces the projectional resolution of identity $\{P_\alpha\}$ ([10], [1]) which served to extend some results from separable Banach spaces to such spaces. S. Trojanski ([11]) used this construction to prove the existence of such $\{P_\alpha\}$ in duals X^* of (WCG) Banach spaces X, where X has a Markuševič basis whose coefficient space is X^*. Here is the system $\{P_\alpha\}$ constructed for spaces with Markuševič basis whose coefficient space is norming. This implies ([13]) that such spaces have an equivalent locally uniformly rotund norm and that the Markuševič basis $\{x_i\}$ contains a basic subset of the same cardinality (cf. Definition 1). If the coefficient space is 1-norming with respect to some Fréchet differentiable norm on X, then X is WCG.

In proving the existence of projections $\{P_\alpha\}$ we follow [1], but the proof of the fundamental lemma 1 [1, Lemma 3] is given here also in a non-convex situation by a slightly different method, which does not use the convexity of Minkowski functional. The needed inequalities are assured on a dense subset and thus the proof follows by continuity arguments. This may be used to carry out the Amir-Lindenstrauss construction of continuous projections also in complete metric (non locally convex) linear spaces which are generated by a weakly compact subset.

Using a theorem of Corson we observe further that WCG and F (= with Fréchet differentiable norm) Banach spaces are Lindelöf in its weak topology (w-Lindelöf). Corson [3] conjectured that Banach space is w-Lindelöf iff it is generated by a weakly compact subset. This proved to be false because of the Rosenthal's example of WCG space which is not hereditary WCG [12]. Rosenthal [12] asks if hereditary WCG spaces are (exactly) the spaces which are w-Lindelöf. Thus our result supports this conjecture, because WCG F spaces are hereditary WCG [7]. For the proof of our result we use the fact that WCG F spaces have a shrinking Markuševič basis (cf. e.g. [7]).

If X is a Banach space, $M \subset X$ and $Y \subset X^*$ then $w(X,Y)\,sp\,M$ (resp. $sp\,M$) denotes the $w(X,Y)$ closed linear (resp. linear) span of M in X. We put also $\overline{sp}\,M = w(X,X^*)\,sp\,M$. Markuševič basis ($M$-basis) of a Banach space X is a system $\{(x_i, x_i^*);\, i \in I\}$

where $x_i \in X$, $x_i^* \in X^*$, $x_i^*(x_j) = \delta_{ij}$, $\overline{sp}\{x_i\} = X$ and $\overline{sp}\{x_i^*\}$ (= the coefficient space) is total on X. $Y \subset X^*$ is called δ-norming, if $\delta \leq \inf_{\|x\|=1}(\sup(f(x);\ f \in Y, \|f\| \leq 1))$. If Y is δ-norming for some $\delta > 0$ then Y is called norming. A Banach space X is locally uniformly rotund (LUR) if, whenever $x_m, x \in X$, $\|x_m\| = \|x\| = 1$, $\lim \|x_m + x\| = 2$, then $\lim \|x_m - x\| = 0$.

We start with

Lemma 1. Let $(N, |\cdot|)$ be a normed space, $Q \subset N^*$ a 1-norming subspace of N^*. Let $N = sp\,K$ where $K \setminus \{0\}$ is a linearly independent subset of N and let K be $w(N, Q)$ compact.

Then, given a sequence $f_1, f_2, \ldots$ in Q and a finite dimensional subspace $B \subset N$, there exists a countable $C \subset K$ with $sp\, C$ containing B, and a linear operator $T : N \longrightarrow N$ such that $|T| = 1$, $TN \subset w(N, Q)\, sp\, C$, $TK \subset K$ for every $z \in N$, $Tb = b$ for every $b \in B$ and $T^* f_k = f_k$ for every $k = 1, 2, \ldots$.

Proof. We may suppose that $B = sp\,(B \cap K)$. For every integer p, let $B_p \subset B$ be finite sets such that $\bigcup B_p$ is dense in B. Similarly let $\Lambda_{mp} \subset R^m$ be finite sets such that $\bigcup_p \Lambda_{mp}$ is dense in R^m.

Now let us fix arbitrary integers m and p and put

$$(1) \qquad H = K^m = K \times K \times \ldots \times K .$$

Then for every $b \in B_p$, every $\lambda \in \Lambda_{mp}$ and every $k = 1, \ldots, p$ we consider the following functions of

$(z_1, \dots, z_m) \in H$:

$$|b + \sum_{i=1}^{m} \lambda_i z_i| \;,\quad f_k(b + \sum_{i=1}^{m} \lambda_i z_i) \;.$$

These $M = card\, B_p \cdot card\, \Lambda_{mp} \cdot (1 + p)$ functions can be regarded as a function $\Phi : H \longrightarrow R^M$ (R^M with a maximal coordinate distance). Using the separability of $\Phi(H) \subset R^M$ we choose a sequence $S = {}^{mp}S\{{}^{mp}x^{\ell}\}_1 = \{({}^{mp}x_1^{\ell}, \dots, {}^{mp}x_m^{\ell})\}_1 \subset H$ such that $\Phi(S)$ is dense in $\Phi(H)$.

Put

$$C = \{{}^{mp}x_j^{\ell} \;;\; m, p, \ell = 1, 2, \dots,\; j = 1, \dots, m\} \cup (B \cap X) .$$

Now, if p is integer, $Z \subset X$, $B \subset Z$, $\dim Z/B = m$, $Z = B \oplus sp\{z_1, \dots, z_m\}$, $(z_1, \dots, z_m) \in H$, we can choose $(x_1, \dots, x_m) \in {}^{mp}S$ such that

$$(2) \qquad |\Phi(x_1, \dots, x_m) - \Phi(z_1, \dots, z_m)| < \frac{1}{p} \;.$$

Let $T_Z^{mp} : Z \longrightarrow C$ be linear mappings defined by

$$T_Z^{mp}(b + \sum_{i=1}^{m} \lambda_i z_i = b + \sum_{i=1}^{m} \lambda_i x_i \;.$$

Put $L_p = \{b + \sum_{i=1}^{m} \lambda_i z_i \;;\; b \in B_p ,\; \lambda \in \Lambda_{mp}\}$ and $L = \bigcup_p L_p$.

If $z \in L_p$ then using (2) we have

$$(3) \quad |T_Z^{mp}(z)| = |b + \Sigma \lambda_i x_i| \le |b + \Sigma \lambda_i z_i| + \frac{1}{p} = |z| + \frac{1}{p} \;.$$

Similarly

$$(4) \qquad |f_k(T_Z^{np}(x)) - f_k(x)| \leq \frac{1}{p} \quad .$$

for all $x \in L_p$ and $k = 1, \dots, p$.

Evidently $T_Z^{np}(K \cap Z) \subset K$ and thus $T_Z^{np}/(K \cap Z) \in K^{K\cap Z}$. By Tychonoff's theorem there is a subnet $\{T_Z^{np}/K \cap Z\}_\alpha$ converging pointwise on $K \cap Z$ and thus on $sp(K \cap Z) = Z$ in the $w(N, Q)$ topology to $T_Z : \longrightarrow w(N, Q) sp\, C$.

Q is 1-norming on $(N, |\cdot|)$ or equivalently the unit ball of $(N, |\cdot|)$ is $w(N, Q)$ closed, or equivalently $|\cdot|$ is lower $w(N, Q)$ semicontinuous. Let $x \in L_q$. Then $|T_Z x| \leq \lim sup_\alpha |T_\alpha^{np_\alpha} x| \leq |x| + \frac{1}{p}$

by (3) and by lower $w(N, Q)$ semicontinuity of $|\cdot|$. Thus, $|T_Z x| \leq |x|$ on L and from the density of $sp\, L$ in Z we have $|T_Z x| \leq |x|$ for every $x \in Z$. Similarly, by (4) and lower $w(N, Q)$semicontinuity of $|\cdot|$ and f_k we have $|f_k(x) - f_k(T_Z x)| = 0$ for all $x \in Z$ and $k = 1, 2, \dots$.

Now as in [1, Lemma 3] the net $\{T_Z\}$ has a $w(N, Q)$ cluster point T . Evidently T has all properties mentioned in our lemma.

Remark. Lemma 1 is listed here in its simplest form and other variants similarly as in [6] may be proved. Other $w(N, Q)$ lower semicontinuous norms or Minkowski functionals on X or its subspaces may be given and projections constructed contractive with respect to them. Some assumptions on norm inequalities may be raised. $w(N, Q)$ closure of any linear independent subset may be preserved.

Notation. In the sequel we will use the following assumptions and notation:

$\{(x_i, x_i^*) ; i \in I\}$ is an $\mathcal{M}$-basis of X with $|x_i| = 1$ for all $i \in I$. We put $K = \{x_i ; i \in I\} \cup \{0\}$, $N = sp\, K$ and $Q = \overline{sp}\{x_i^* ; i \in I\}$. If $0 \notin M \subset K$ then we put $M^* = \{x_i^* ; x_i \in M\}$, if $x = x_i \in K$ then we put $x^* = x_i^*$.

Lemma 2. K is $w(X, Q)$ compact.

Proof: The $w(X, Q)$ topology and $w(X, sp\{x_i\})$ topology coincide on K because K is bounded. Let $\{x_{i_\alpha}\}$ be a net in K and $i \in I$. Then $x_i^*(x_{i_\alpha}) \longrightarrow 0$ if $i_\alpha \neq i$ and thus $x_{i_\alpha} \longrightarrow 0$ in $w(X, sp\{x_i\})$ topology.

Lemma 3. Let $C \subset K$. Then $N \cap w(N, Q)\, sp\, C = N \cap \overline{sp}\, C = sp\, C$.

Proof: It is easy to see that all subspaces of the form $sp\, C$ are $w(N, Q)$ closed in N.

Lemma 4. Let X, N, Q, K be as in Notation and let Q be 1-norming on $(X, |\cdot|)$. Then, given a finite subset $L \subset K$, $(0 \notin L)$, there exists a countable set $C \subset K$ and a linear operator $T : N \longrightarrow w(N, Q)\, sp\, C = sp\, C$ with $|T| \leq 1$, $TK \subset K$ for all $z \in N$, $Tb = b$ for every $b \in L$ and $x^*(Tz) = x^*(z)$ for every $x \in L$ and $z \in N$.

Proof: With our notation we have on $(N, |\cdot|)$ exactly the situation of Lemma 1. We also put $B = sp\, L$ and $f_i = x_i^*$ for $x_i \in L$.

Lemma 5. Let N, Q, K be as in Lemma 4, m an infinite cardinal number and $L \subset K \setminus \{0\}$ with $card\, L = m$.

Then there exists a subset $C \subset K \setminus \{0\}$ with $L \subset C$, $\operatorname{card} C = m$ and a projection $P: N \to N$ with $|P| = 1$, $PN = \operatorname{sp} C$, $PK \subset K$ and $x^*(Px) = x^*(x)$ for all $x \in N$ and all $x^* \in C^*$.

Proof: Similarly as in the proof of Lemma 4 in [1] we use the transfinite induction on m. Assume $m = \aleph_0$. As in [1], we define inductively a sequence of countable sets $C_m = \{x_{mi}\}_i \subset K \setminus \{0\}$ and linear operators $T_m : N \to N$ with $|T_m| = 1$, $T_m K \subset K$, $T_m N \subset C_m$, $T_m(x_{ji}) = x_{ji}$ and $x^*_{ji}(Tx) = x^*_{ji}(x)$ for $j = 0, \dots, m-1$ and $i = 1, \dots, m$. Put $C = \bigcup C_j$. Let $P: N \to N$ be a $w(N, Q)$ cluster point of the sequence $\{T_m\}$. Using Lemma 3 we have $\operatorname{sp} C \subset PN \subset w(N, Q) \operatorname{sp} C = \operatorname{sp} C$, showing that $PN = \operatorname{sp} C$ and thus P is a projection. Further, the inductive proof follows exactly as the proof of Lemma 4 in [1]. The cluster points are here in the $w(N, Q)$ topology and Lemma 3 is used.

Remark. The projection P is determined by the properties: $PN = \operatorname{sp} C$ and $x^*(Px) = x^*(x)$ for $x \in C$ and $x \in N$. Indeed, if $x \in K \setminus C$, then $Px = 0$.

Proposition 1. Let X, N, Q, K be as in Lemma 4. Let ξ be the first ordinal of cardinality $\operatorname{card} K$ ($= \operatorname{card} I$) and let $\{x_\alpha ; \alpha < \xi\} = K$. Then there exists a "long sequence" of linear projections $\{P_\alpha ; \omega \le \alpha \le \xi\}$ of X and subsets $C_\alpha \subset K$ satisfying $|P_\alpha| = 1$, $x_\alpha \in P_{\alpha+1} X = \overline{\operatorname{sp}}\, C_{\alpha+1}$, $\operatorname{card} C_\alpha = \overline{\overline{\alpha}}$, $C_\beta \subset C_\alpha$ whenever $\beta < \alpha$, $C^*_\alpha \subset P^*_\alpha X$, $\bigcup_{\beta<\alpha} P_{\alpha+1}$ is norm dense in

$P_\alpha X$ for every $\alpha > \omega$ and $P_\alpha X = \overline{sp}\, C_\alpha$.

Thus by the above remark $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ whenever $\beta < \alpha$. For every $x \in X$ and every $\varepsilon > 0$, the set $\{\alpha ; |P_{\alpha+1} x - P_\alpha x| > \varepsilon\}$ is finite. $P_\alpha \neq P_\beta$ if $\alpha \neq \beta$.

Proof: Similarly as in the proof of Lemma 6 in [1] we construct (using Lemmas 3 and 5) such a "sequence" of projections $\{P_\alpha ; \omega \leq \alpha \leq \xi\}$ of N . Evidently they can be extended to projections $P_\alpha : N \longrightarrow X$ with $|P_\alpha| = 1$. The last assertion is proved similarly as Lemma 7 in [1] using equicontinuity of P_α's and Lemma 3 follows from [7, Lemma 2] .

Definition 1 (Bessaga). Let ξ be an ordinal number. (Orthogonal) projectional basis of type ξ is a system of projections $\{P_\alpha ; \alpha \leq \xi\}$ such that

$sup|P_\alpha| < \infty, (|P_\alpha| = 1), dim(P_{\alpha+1} - P_\alpha)X = 1, P_\alpha P_\beta = P_\beta \ P_\alpha = P_\beta$

for $\beta < \alpha$ and the function $\alpha \longrightarrow P_\alpha x$ is norm continuous on ordinals for every $x \in X$.

$\{x_\alpha ; \alpha < \xi\}$ where $x_\alpha \in (P_{\alpha+1} - P_\alpha)X$ is then called the basis of X . A system $\{y_\alpha ; \alpha < \xi\}$ of elements of X is called a basic sequence if it is a basis of $\overline{sp}\{y_\alpha\}$.

Proposition 2. Let $(X, |\cdot|)$ (X non-separable) have M-basis $\{(x_i, x_i^*)\}$, whose coefficient space is 1-norming. Then there is orthogonal basic sequence $\{y_\alpha\} \subset \{x_i\}$ with $card\{y_\alpha\} = card\{x_i\} = dens\, X$.

Proof: If $\{P_\alpha ; \omega \le \alpha \le \xi\}$ is a system of projections from Proposition 1, choose $y_\alpha \in (P_{\alpha+1} - P_\alpha)X \cap \{x_i\}$. Evidently $\{Q_\alpha ; \omega \le \alpha \le \xi\}$ where $Q = P_\alpha / \overline{sp}\{y_\alpha\}$ is an orthogonal projectional basis of $\overline{sp}\{y_\alpha\}$ (cf. also [11]).

Proposition 3. Let $(X, |\cdot|)$ be a Banach space with Fréchet smooth norm. If $(X, |\cdot|)$ has an M-basis whose coefficient space is 1-norming, then X is WCG.

Proof: By Proposition 1, Lemma 3 of [7] and similarly as in the proof of Lemma 4 of [7], X has an M-basis whose coefficient space is X^* and thus X is WCG by Lemma 2.

Theorem. Let X be a Banach space with an M-basis whose coefficient space Q is norming. Then X admits an equivalent LUR norm which is lower $w(X, Q)$ semicontinuous.

Proof: First we introduce an equivalent norm $\|x\| = \sup\{f(x); f \in Q, |f| \le 1\}$ for which Q is 1-norming. Starting with Proposition 1, we proceed exactly as in the proof of Theorem 1 in [13], p. 177-178. To show that operators $T_\alpha : X \longrightarrow X$ satisfying (i) - (iii) ([Prop. 1, p.175]) exist, we proceed by induction on cardinality of M-basis, noting that $(P_{\alpha+1} - P_\alpha)X$ has an M-basis with norming coefficients.

The one-to-one continuous linear operator of X into $c_0(\Gamma)$ is provided by a theorem of J. Dyer [5].

Remark. It was shown by J. Lindenstrauss that every WCG space has an $\mathcal{M}$-basis. Every separable space has an $\mathcal{M}$-basis with norming coefficient space [9]. This, together with the Theorem suggest the following questions:

1) Has every WCG space an $\mathcal{M}$-basis with norming coefficient space?

2) Does every space with an $\mathcal{M}$-basis admit an equivalent LUR norm?

Remark. If the WCG space X has an $\mathcal{M}$-basis with a norming coefficient space, then similarly as in [8, corollary 1 and Lemma 6] we see that X has Gâteaux smooth partitions of unity (subordinated to any open covering).

Now we show that WCG F spaces are w-Lindelöf. For this we recall (cf. [3], p. 2) that a subset $A \subset c_0(\Gamma)$ is said to be almost invariant under projections if there is a collection $\{\Gamma\sigma ; \sigma \in \Sigma\}$ of countable subsets of Γ such that each countable subset of Γ is contained in one of the $\Gamma\sigma$, $\Gamma\sigma_1 \subset \Gamma\sigma_2 \subset \Gamma\sigma_3 \ldots$ implies that $\bigcup_{i=1}^{\infty} \Gamma\sigma_i$ is one of $\Gamma\sigma$, and such that $A/\Gamma\sigma \subset A$ for every $\sigma \in \Sigma$. Here $A/\Gamma\sigma = \{a/\Gamma_\sigma ; a \in A\}$ and a/Γ_σ is the element of $c_0(\Gamma)$ which agrees with a on Γ_σ and which has the value 0 for $\gamma \in \Gamma \setminus \Gamma_\sigma$.

Lemma 6. Let X be a Banach space with an $\mathcal{M}$-basis $\{(x_i, f_i) ; i \in \Gamma\}$ whose coefficient space is 1-norming and suppose that $\|f_i\| = 1$. Thus $Tx = \{f_i(x); i \in \Gamma\}$ defines a continuous linear mapping of X into $c_0(\Gamma)$. Let B be the closed unit ball of X. Then TB is a subset of $c_0(\Gamma)$ which is almost invariant under projections.

Proof: Let Σ be the set of all projections σ : $X \longrightarrow X$ with a separable range given by Lemma 5. Then $\sigma X = \overline{sp}\{x_i ; i \in \Gamma_\sigma\}$ where $\Gamma_\sigma \subset \Gamma$ is countable. Evidently the collection $\{\Gamma_\sigma ; \sigma \in \Sigma\}$ has the required properties. (If $\Gamma_{\sigma_1} \subset \Gamma_{\sigma_2} \subset \dots$, then $\lim \sigma_n = \sigma$ exists and $\Gamma_\sigma = \bigcup \Gamma_{\sigma_i}$. If $x \in B$ then $Tx/\Gamma_\sigma = T\sigma x$ for $\sigma \in \Sigma$, showing that $TB/\Gamma_\sigma \subset TB$.)

Corollary (Corson). Let X be a Banach space with an M-basis (x_i, f_i) whose coefficient space $Y = \overline{sp}\{f_i\}$ is norming. Then X is Lindelöf in the $w(X,Y)$-topology. Especially, every space with a shrinking M-basis (i.e. $\overline{sp}\{f_i\} = X^*$) is Lindelöf in the w-topology.

Proof: We may assume that Y is 1-norming. It suffices to prove that B is $w(X,Y)$-Lindelöf or $w(X, sp\{f_i\})$-Lindelöf. Evidently $T: X \longrightarrow c_0(\Gamma)$ defined above is the homeomorphism with respect to $w(X, sp\{f_i\})$-topology on X and the topology of coordinate convergence on $c_0(\Gamma)$. But the latter is Lindelöf on every subset almost invariant under projections by the theorem of Corson [3, Lemma 1].

Remark. Similarly, following Corson and Lindenstrauss, Theorem 2.4 of [4] still holds if H is a reflexive Banach space: Let X be a topological space which is a continuous image of a separable metric space, and let H^w be any reflexive Banach space in the weak topology. Then $C(X, H^w)$ is Lindelöf in the topology of pointwise convergence.

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