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NONLINEAR OPERATOR EQUATIONS AND BOUNDARY-VALUE PROBLEMS

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Abstract: Let W and V be real Banach spaces with duals W^* and V^*, respectively. Suppose that $W \subset V$ and let I_1 be the injection mapping of W into V. Let T be a mapping from $D_T \subset V$ into W^* and $f \in V^*$. Under suitable conditions on T the existence of at least one solution $u_0 \in D_T$ of

$$Tu = I_1^* f$$

is proved using regularization methods, where I_1^* is the dual mapping of I_1. An application to nonlinear elliptic boundary-value problems is given.

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_. Let W and V be two real Banach spaces with duals W^* and V^*, respectively, and let $W \subset V$. Recently, the author [6] has studied mappings T with domain of definition D_T in V and range in V^*. Under suitable conditions on T there exists at least one solution $u_0 \in D_T$ to $Tu = f$ with $f \in V^*$. The proof is based on regularization methods. This general existence theorem was applied to nonlinear elliptic boundary-value problems of

2. See also Hess [4], where a related theorem is given using other regularization methods. It is an open question, whether the general existence theorem in [6] can also be applied to differential equations of order greater than 2.

In this paper, we will suppose that T is a mapping with domain of definition D_T in V and range in W^* and we will prove the existence of a solution $u_0 \in D_T$ to $Tu = I_1^* f$, where I_1^* is the dual mapping of the injection mapping I_1 of W into V (Theorem 1). This theorem is applied to a class of nonlinear elliptic boundary value problems of order $2m$ (Theorem 2). Bui An Ton [3] also studied operator equations of the form $Tu = I_1^* f$, but he assumes that T is a mapping of $D_T = V$ into W^*. It should be remarked, that the result of [6] is valid - in the case of equations of order 2 - also for more general classes of differential equations, but for the class of differential equations studied in this paper, our present result is (for $m = 1$) less restrictive than the result of [6].

2. Let V and W be two real reflexive separable Banach spaces with $W \subset V$, where the natural injection mapping I_1 of W into V shall be continuous.

Let V^* and W^* be the duals of V and W, respectively. The pairing between V and V^* shall be denoted by $((\cdot,\cdot))$ and that of W and W^* by $(\cdot,\cdot)$.

By $\longrightarrow$ and $\rightharpoonup$ we will denote the strong and weak convergence, respectively.

By a theorem of Browder-Bui An Ton [2], there exists a separable Hilbert space H (the inner product shall be denoted by $\langle \cdot , \cdot \rangle$) and a compact linear mapping I_2 of H into W such that $I_2(H)$ is dense in W (see also [6]).

The dual mappings of I_2 and I_1 shall be denoted by I_2^* and I_1^*, respectively.

To prove an existence theorem for operator equations with mappings from $D \subset V$ into W^* we will use regularization methods. Therefore we introduce

<u>Assumption 1</u>: (a) For each $\varepsilon \in]0,1]$, let $A(\varepsilon,\cdot):$ $: V \longrightarrow V^*$ be bounded (i.e. maps bounded sets into bounded sets) and demi-continuous (i.e. continuous from the strong to the weak topology).

(b) Let there exist a mapping $\mathcal{A}: V \longrightarrow W^*$ such that any sequences $\{w_n\} \subset W$ and $\{\varepsilon_n\} \subset]0,1]$ satisfying $\varepsilon_n \longrightarrow 0$ and $I_1 w_n \longrightarrow u_0$ in V imply

$$I_1^* A(\varepsilon_n, I_1 w_n) \longrightarrow \mathcal{A}(u_0) \text{ in } W^* .$$

(c) Suppose that for all $\varepsilon \in]0,1]$ and all $w \in W$

$$((A(\varepsilon, I_1 w), I_1 w)) \geq \varphi(\| I_1 w \|_V) \cdot \| I_1 w \|_V$$

where $\varphi: \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1$ with (i) $\varphi(x)$ continuous; (ii) $\varphi(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

<u>Assumption 2</u>: (a) For each $\varepsilon \in]0,1]$ let $B(\varepsilon,\cdot):$ $: V \longrightarrow V^*$ be bounded and demicontinuous. Furthermore

suppose that for all $\varepsilon \in]0,1]$ and all $w \in W$

$$((B(\varepsilon, I_1 w), I_1 w)) \geq 0 .$$

(b) Let there exist a mapping $B : D(B) \subset V \longrightarrow W^*$, i.e. $D(B) = \{u \in V : B(u) \in W^*\}$, such that any sequences $\{\varepsilon_n\} \subset]0,1]$ and $\{w_n\} \subset W$ satisfying $\varepsilon_n \longrightarrow 0$, $I_1 w_n \longrightarrow u_0$ in V and $0 \leq ((B(\varepsilon_n, I_1 w_n), I_1 w_n)) \leq \mathscr{C}$ with some $\mathscr{C} > 0$ imply $u_0 \in D(B)$, i.e. $B(u_0) \in W^*$, and

$$I_1^* B(\varepsilon_n, I_1 w_n) \longrightarrow B(u_0) \quad \text{in } W^* .$$

In this section we will consider the existence of a solution $u_0 \in D(B)$ to

$$A(u) + B(u) = I_1^* f \tag{2.1}$$

with $f \in V^*$.

We formulate our main theorem:

<u>Theorem 1</u>: Suppose that Assumption 1,2 holds. Let $f \in V^*$. Then there exists at least one solution $u_0 \in D(B)$ satisfying (2.1).

Proof: The proof follows by several steps.

(a) For $\varepsilon \in]0,1]$ and $x \in H$ we set

$$T(\varepsilon, x) := \frac{1}{\varepsilon}(I_2^*, I_1^* f - I_2^* I_1^* A(\varepsilon, I_1 I_2 x) - I_2^* I_1^* B(\varepsilon, I_1 I_2 x)) .$$

By Assumption 1 (a), 2 (a) and the above remarks, it follows that for each $\varepsilon \in]0,1]$, the mapping $T(\varepsilon, \cdot)$ is compact and continuous from H to H. Furthermore it follows by Assumptions 1 (c), 2 (a)

$$\langle x - T(\varepsilon, x), x\rangle = \|x\|_H^2 - \frac{1}{\varepsilon}\langle I_2^* I_1^* f, x\rangle +$$

$$+\frac{1}{\varepsilon}\{\langle I_2^* I_1^* A(\varepsilon, I_1 I_2 x), x\rangle + \langle I_2^* I_1^* B(\varepsilon, I_1 I_2 x), x\rangle\}$$

$$= \|x\|_H^2 - \frac{1}{\varepsilon}((f, I_1 I_2 x)) + \frac{1}{\varepsilon}\{((A(\varepsilon, I_1 I_2 x)), I_1 I_2 x)) +$$

$$+ ((B(\varepsilon, I_1 I_2 x), I_1 I_2 x)) \geq \|x\|_H^2 - \frac{1}{\varepsilon}\|f\|_{V*}\|I_1 I_2 x\|_V +$$

$$+\frac{1}{\varepsilon}\varphi(\|I_1 I_2 x\|_V)\|I_1 I_2 x\|_V \geq \|x\|_H(\|x\|_H +$$

$$+\frac{1}{\varepsilon}\frac{\|I_1 I_2 x\|_V}{\|x\|_H}(\varphi(\|I_1 I_2 x\|_V) - \|f\|_{V*})) \geq 0$$

for all $x \in S_R : \{x \in H : \|x\|_H = R_\varepsilon\}$, where R_ε is a suitable positive constant. Indeed, this follows by the assumption on φ and the inequality $\|I_1 I_2 x\|_V \leq \gamma \|x\|_H$ with some constant $\gamma > 0$. Hence by a theorem of Krasnoselskii there exists for each $\varepsilon \in]0,1]$ a fixed point $x_\varepsilon \in H$ of $T(\varepsilon, \cdot)$, i.e.

$$(2.2)\quad \varepsilon \cdot x_\varepsilon = \varepsilon \cdot T(\varepsilon, x_\varepsilon) = I_2^* I_1^* f - I_2^* I_1^* A(\varepsilon, I_1 I_2 x_\varepsilon) - I_2^* I_1^* B(\varepsilon, I_1 I_2 x_\varepsilon).$$

Therefore by Assumption 1 (c), 2 (b)

$$(2.3)\begin{cases} 0 = \langle \varepsilon x_\varepsilon + I_2^* I_1^* A(\varepsilon, I_1 I_2 x_\varepsilon) + I_2^* I_1^* B(\varepsilon, I_1 I_2 x_\varepsilon) - \\ - I_2^* I_1^* f, x_\varepsilon\rangle = \varepsilon \|x_\varepsilon\|_H^2 + ((A(\varepsilon, I_1 I_2 x_\varepsilon), I_1 I_2 x_\varepsilon)) + \\ + ((B(\varepsilon, I_1 I_2 x_\varepsilon), I_1 I_2 x_\varepsilon)) - ((f, I_1 I_2 x_\varepsilon)) \geq \varepsilon \|x_\varepsilon\|_H^2 + \\ + (\varphi(\|I_1 I_2 x_\varepsilon\|_V) - \|f\|_{V*}) \cdot \|I_1 I_2 x_\varepsilon\|_V . \end{cases}$$

Hence there exist positive constants $\mathscr{C}_1, \mathscr{C}_2$ such that

$$(2.4)\qquad \sqrt{\varepsilon}\|x_\varepsilon\|_H \leq \mathscr{C}_1, \quad \|I_1 I_2 x_\varepsilon\|_V \leq \mathscr{C}_2$$

for all $\varepsilon \in]0,1]$.

(b) By virtue of (2.4) there exists a sequence $\{\varepsilon_m\} \subset]0,1]$ such that $\varepsilon_m \to 0$, $\varepsilon_m \cdot x_{\varepsilon_m} \to 0$ in H and $I_1 I_2 x_{\varepsilon_m} \rightharpoonup u_0$ in V. From (2.3) we obtain by Assumption 1 (c), 2 (a)

$$0 \le ((B(\varepsilon_m; I_1 I_2 x_{\varepsilon_m}), I_1 I_2 x_{\varepsilon_m})) = -\varepsilon_m \|x_m\|_H^2$$

$$-((A(\varepsilon_m, I_1 I_2 x_{\varepsilon_m}), I_1 I_2 x_{\varepsilon_m})) + ((f, I_1 I_2 x_{\varepsilon_m}))$$

$$\le -\varphi(\|I_1 I_2 x_{\varepsilon_m}\|_V)\|I_1 I_2 x_{\varepsilon_m}\|_V + \|f\|_{V*}\|I_1 I_2 x_{\varepsilon_m}\|_V \le \mathcal{C}$$

with some constant $\mathcal{C}$ by (2.4) and the assumption on φ. Hence by Assumption 1 (b), 2 (b) and $w_m := I_2 x_{\varepsilon_m} \in W$, we obtain $u_0 \in D(B)$ and

$$I_1^* A(\varepsilon_m, I_1 I_2 x_{\varepsilon_m}) \rightharpoonup A(u_0); \; I_1^* B(\varepsilon_m, I_1 I_2 x_{\varepsilon_m}) \rightharpoonup B(u_0)$$

in W^* as $m \to \infty$. From (2.2) it follows for each $x \in H$

$$\langle \varepsilon_m x_{\varepsilon_m}, x\rangle + (I_1^* A(\varepsilon_m, I_1 I_2 x_{\varepsilon_m}), I_2 x) +$$
$$+ (I_1^* B(\varepsilon_m, I_1 I_2 x_{\varepsilon_m}), I_2 x) = (I_1^* f, I_2 x) .$$

Therefore as $m \to \infty$

$$(A(u_0), I_2 x) + (B(u_0), I_2 x) = (I_1^* f, I_2 x)$$

from which we get

$$A(u_0) + B(u_0) = I_1^* f ,$$

since $I_2(H)$ is dense in W, proving Theorem 1.

Remark: Theorem 1 in this section is related to Theorem 1 in [6] but none of them implies the other (see the following application).

3. In this section we will apply Theorem 1 to nonlinear elliptic differential equations. We use the notations of Browder in [1]. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open domain with sufficiently smooth boundary $\partial\Omega$ such that the Imbedding Theorems of Sobolev are applicable (see e.g. [1]).

It is our purpose to study differential equations of the form

$$\sum_{|\alpha|=m} (-1)^{|\alpha|} D^\alpha (A_\alpha(x, \xi_{m-1}(u)(x)) D^\alpha u(x))$$

$$+ \sum_{|\alpha| \le m-1} (-1)^{|\alpha|} D^\alpha B_\alpha(x, \xi_{m-1}(u)(x)) = f(x)$$

for $x \in \Omega$ with Dirichlet boundary conditions. Precisely, we set $[f, g] := \int_\Omega f(x) g(x) dx$, $V := \overset{\circ}{W}_{m,2}(\Omega)$ and $W := W_{m^*,2}(\Omega) \cap V$ with $m^* > m + n/2$, where we introduce in W the norm of $W_{m^*,2}(\Omega)$. In addition, we set

$$(3.1)\quad D(B) := \{ u \in V : | \sum_{|\alpha| \le m-1} [B_\alpha(\cdot, \xi_{m-1}(u)), D^\alpha w] | \le$$

$$\le \mathscr{C}_u \| w \|_W$$

for all $w \in W$ with some constant $\mathscr{C}_u$ } .

Furthermore let f be an element of V^* . Then we ask for an element $u_0 \in D(B)$ satisfying

$$(3.2)\quad \begin{cases} \sum_{|\alpha|=m} [A_\alpha(\cdot, \xi_{m-1}(u_0)) D^\alpha u_0, D^\alpha w] \\ \quad + \sum_{|\alpha| \le m-1} [B_\alpha(\cdot, \xi_{m-1}(u_0)), D^\alpha w] = [f, w] \end{cases}$$

for all $w \in W$.

Assumption 3: (a) Each $A_\alpha(x, \xi_{m-1})$ (with $|\alpha| = m$) is measurable in x for fixed ξ_{m-1} in $\mathbb{R}^{s_{m-1}}$ and continuous in ξ_{m-1} on $\mathbb{R}^{s_{m-1}}$ for almost all x in Ω. Let k be the greatest integer less than $m - n/2$ and let ξ_k denote the vector $\xi_k := \{\xi_\alpha : |\alpha| \le k\}$ from the vector space $\mathbb{R}^{s_k}$. There exist continuous functions c_α and c_1 from $\mathbb{R}^{s_k}$ to $L^2(\Omega)$ and $\mathbb{R}^1$, respectively, and a constant $c_0 > 0$, such that the following inequalities hold:

$$c_0 \le A_\alpha(x, \xi_{m-1}) \le c_\alpha(\xi_k)(x) + c_1(\xi_k) \sum_{k<|\beta|\le m-1} |\xi_\beta|^{\frac{t_\beta}{2}}$$

for all $x \in \Omega$ and $\xi_{m-1} \in \mathbb{R}^{s_{m-1}}$ with

$$\frac{1}{t_\beta} > \frac{1}{2} - \frac{1}{n}(m - |\beta|) .$$

(b) Each $B_\alpha(x, \xi_{m-1})$ (with $|\alpha| \le m-1$) is a continuous function from $\Omega \times \mathbb{R}^{s_{m-1}}$ to $\mathbb{R}^1$ such that for all $\varepsilon \in [0,1]$, all ξ_{m-1} in $\mathbb{R}^{s_{m-1}}$ and almost all x in Ω

$$\sum_{|\alpha|\le m-1} \frac{B_\alpha(x, \xi_{m-1})\,\xi_\alpha}{1 + \varepsilon\,|B_\alpha(x, \xi_{m-1})|} \ge 0 .$$

Suppose that there exists a constant $c_2 \ge 0$ and a function $F : \Omega \times \mathbb{R}^{s_{m-1}} \times \mathbb{R}^{s_{m-1}} \longrightarrow \mathbb{R}^1$ such that for all $\varepsilon \in [0,1]$, all ξ_{m-1}, ξ'_{m-1} in $\mathbb{R}^{s_{m-1}}$ and almost all x in Ω

$$\left|\sum_{|\alpha|\le m-1} \frac{B_\alpha(x, \xi_{m-1})\,\xi'_\alpha}{1 + \varepsilon\,|B_\alpha(x, \xi_{m-1})|}\right| \le F(x, \xi_{m-1}, \xi'_{m-1}) + c_2 \sum_{|\alpha|\le m-1} \frac{B_\alpha(x, \xi_{m-1})\,\xi_\alpha}{1 + \varepsilon\,|B_\alpha(x, \xi_{m-1})|} .$$

In addition, suppose that for all $w \in W$, the mapping $F(\xi_{m-1}(\cdot), \xi_{m-1}(w))$, defined by

$F(\xi_{m-1}(u), \xi_{m-1}(w))(x) := F(x, \xi_{m-1}(u)(x), \xi_{m-1}(w)(x))$,

is bounded and continuous from $W_{m-1,2}$ to L^1.

<u>Theorem 2</u>: Let Assumption 3 be satisfied. Then there exists at least one solution $u_0 \in D(B)$ of (3.2).

Proof: (a) We remark that by the Imbedding Theorems of Sobolev it follows

$$W_{m^*,2}(\Omega) \subset C^m(\overline{\Omega}); \quad W_{m^*,2}(\Omega) \subset W_{m,2}(\Omega)$$

with continuous injection. Hence W and V are two real reflexive separable Banach spaces with $W \subset V$, where the injection mapping of W into V is continuous.

(b) We now define for $\varepsilon \in]0,1]$ and $u, v \in V$

$$a(\varepsilon, u, v) := \sum_{|\alpha|=m} \left[\frac{A_\alpha(\cdot, \xi_{m-1}(u))}{1+\varepsilon A_\alpha(\cdot, \xi_{m-1}(u))} D^\alpha u, D^\alpha v\right],$$

$$b(\varepsilon, u, v) := \sum_{|\alpha| \leq m-1} \left[\frac{B_\alpha(\cdot, \xi_{m-1}(u))}{1+\varepsilon |B_\alpha(\cdot, \xi_{m-1}(u))|}, D^\alpha v\right].$$

For $u \in V$ and $w \in W$ let

$$a(u, w) := \sum_{|\alpha|=m} [A_\alpha(\cdot, \xi_{m-1}(u)) D^\alpha u, D^\alpha w]$$

and for $u \in D(B)$ and $w \in W$ we set

$$b(u, w) := \sum_{|\alpha| \leq m-1} [B_\alpha(\cdot, \xi_{m-1}(u)), D^\alpha w].$$

It follows by Assumption 3 and a well-known theorem (see e.g. [1]): For each $\varepsilon \in]0,1]$ there exist bounded continuous mappings $A(\varepsilon,\cdot): V \longrightarrow V^*$ and $B(\varepsilon,\cdot): V \longrightarrow V^*$ such that for all $u, v \in V$

$$((A(\varepsilon,u),v)) = a(\varepsilon,u,v), \quad ((B(\varepsilon,u),v)) =$$
$$= b(\varepsilon,u,v) .$$

By (3.1) it follows the existence of a mapping $B: D(B) \longrightarrow W^*$ such that for all $u \in D(B)$ and $w \in W$ we have

$$(B(u),w) = b(u,w) .$$

By the Imbedding Theorem of Sobolev $W_{m,2}(\Omega) \subset W_{|\beta|,t_\beta}(\Omega)$ for $b < |\beta| \le m-1$ and $W_{m,2}(\Omega) \subset C^{|\beta|}(\bar{\Omega})$ for $|\beta| \le b$ with continuous and compact injection mapping. Hence any weakly convergent sequence in $W_{m,2}(\Omega)$ is strongly convergent in $W_{|\beta|,t_\beta}(\Omega)$ for $b < |\beta| \le m-1$ and in $C^{|\beta|}(\bar{\Omega})$ for $|\beta| \le b$, respectively. Since $W \subset C^m(\bar{\Omega})$, we obtain by Assumption 3(a) for $u \in V$ and $w \in W$ using the inequality of Schwarz

$$\begin{aligned}
&\Big|\sum_{|\alpha|=m} [A_\alpha(\cdot,\xi_{m-1}(u))D^\alpha u, D^\alpha w]\Big| \\
&\quad \le \sum_{|\alpha|=m} \|D^\alpha w\|_{C^0} \, \|A_\alpha(\cdot,\xi_{m-1}(u))D^\alpha u\|_{L^1} \\
&\quad \le \sum_{|\alpha|=m} \|D^\alpha w\|_{C^0} \, \|A_\alpha(\cdot,\xi_{m-1}(u))\|_{L^2} \cdot \|D^\alpha u\|_{L^2} \\
&\quad \le \mathscr{C}_1(\|u\|_V) \sum_{|\alpha|=m} \|D^\alpha w\|_{C^0} \le \mathscr{C}_2(\|u\|_V)\, \|w\|_W ,
\end{aligned}$$

i.e. there exists a mapping $A: V \longrightarrow W^*$ such that for all $u \in V$ and all $w \in W$

$$(Au,w) = a(u,w) .$$

(c) We now apply Theorem 1. We first prove Assumption 1 (b). Let $\{w_n\} \subset W$ and $\{\varepsilon_n\} \subset]0,1]$ with $\varepsilon_n \to 0$ and $u_n := I_* w_n \longrightarrow u_0$ in V. Then by the remark under (b)

$\{D^\beta u_m\}$ converges strongly to $D^\beta u_0$ in $L^{t_\beta}(\Omega)$ for $\ell < |\beta| \le m-1$ and in $C^0(\overline{\Omega})$ for $|\beta| \le \ell$, respectively. Let $w \in W$ then we obtain by $W \subset C^m(\overline{\Omega})$ and Assumption 3

$$|(I_1^* A(\varepsilon_m, u_m), w) - (A(u_0), w)| = |(I_1^* A(\varepsilon_m, u_m) - A(u), w)|$$
$$= \sum_{|\alpha|=m} \left|\left[\frac{A_\alpha(\cdot, \xi_{m-1}(u_m))}{1+\varepsilon_m A_\alpha(\cdot, \xi_{m-1}(u_m))} D^\alpha u_m - A_\alpha(\cdot, \xi_{m-1}(u_0)) D^\alpha u_0, D^\alpha w\right]\right|$$
$$\le \mathcal{J}_{1m} + \mathcal{J}_{2m}$$

with

$$\mathcal{J}_{1m} := \sum_{|\alpha|=m} |[A_\alpha(\cdot, \xi_{m-1}(u_0)) D^\alpha w, D^\alpha u_m - D^\alpha u_0]|$$
$$\mathcal{J}_{2m} := \sum_{|\alpha|=m} \left|\left[\left(\frac{A_\alpha(\cdot, \xi_{m-1}(u_m))}{1+\varepsilon_m A_\alpha(\cdot, \xi_{m-1}(u_m))} - A_\alpha(\cdot, \xi_{m-1}(u_0))\right) D^\alpha w, D^\alpha u_m\right]\right|.$$

It follows by virtue of $A_\alpha(\cdot, \xi_{m-1}(u_0)) D^\alpha w \in L^2(\Omega)$ and $D^\alpha u_m \longrightarrow D^\alpha u_0$ in $L^2(\Omega)$ for $|\alpha| = m$ that $\mathcal{J}_{1m} \longrightarrow 0$ as $n \longrightarrow \infty$.

By the inequality of Schwarz and $w \in C^m(\overline{\Omega})$ it follows

$$\mathcal{J}_{2m} \le \sum_{|\alpha|=m} \|D^\alpha u_m\|_{L^2} \left\|\left(\frac{A_\alpha(\cdot, \xi_{m-1}(u_m))}{1+\varepsilon_m A_\alpha(\cdot, \xi_{m-1}(u_m))} - \right.\right.$$
$$\left.\left. - A_\alpha(\cdot, \xi_{m-1}(u_0))\right) D^\alpha w\right\|_{L^2}$$
$$\le \sum_{|\alpha|=m} \|D^\alpha u_m\|_{L^2} \|D^\alpha w\|_{C^0} \left\|\frac{A_\alpha(\cdot, \xi_{m-1}(u_m))}{1+\varepsilon_m A_\alpha(\cdot, \xi_{m-1}(u_m))} - A_\alpha(\cdot, \xi_{m-1}(u_0))\right\|_{L^2}$$
$$\le \mathcal{C} \|u_m\|_V \|w\|_W \sum_{|\alpha|=m} (\|A_\alpha(\cdot, \xi_{m-1}(u_m)) - A_\alpha(\cdot, \xi_{m-1}(u_0))\|_{L^2}$$
$$+ \operatorname*{ess\,sup}_{x\in\Omega} \left|1 - \frac{1}{1+\varepsilon_m A_\alpha(x, \xi_{m-1}(u_m)(x))}\right| \cdot \|A_\alpha(\cdot, \xi_{m-1}(u_m))\|_{L^2})$$

with some $\mathcal{C} > 0$. By Assumption 3 (a) and the above remark the right hand side of the inequality converges to zero as $m \to \infty$. Hence it follows $I_1^* A(\varepsilon_m, u_m) \longrightarrow A(u_0)$ in W^*, proving Assumption 1 (b).

(d) Let $\varepsilon \in]0,1]$ and $w \in W$, then we obtain by Assumption 3 (a)

$$((A(\varepsilon, I_1 w), I_1 w)) = \sum_{|\alpha|=m} [\frac{A_\alpha(\cdot, \xi_{m-1}(w))}{1+\varepsilon A_\alpha(\cdot, \xi_{m-1}(w))} D^\alpha w, D^\alpha w]$$

$$\geq \sum_{|\alpha|=m} [\frac{c_0}{1+\varepsilon c_0} D^\alpha w, D^\alpha w] = \frac{c_0}{1+\varepsilon c_0} \sum_{|\alpha|=m} \| D^\alpha w \|^2_{L^2} \geq c_1 \| I_1 w \|^2_V$$

with some $c_1 > 0$, since for $u \in V$ the usual norm $\| u \|_V$ of V and $(\sum_{|\alpha|=m} \| D^\alpha u \|^2_{L^2})^{\frac{1}{2}}$ are equivalent norms (see [5]). Hence Assumption 1 (c) is satisfied.

(e) The second part of Assumption 2 (a) is a direct consequence of Assumption 3 (b), while Assumption 2 (b) is proved in [6] by using Assumption 3 (b).
Hence Theorem 2 follows by Theorem 1.

<u>Remark</u>: (a) Conditions on $B_\alpha(x, \xi_{m-1})$ which are more useful in applications and which imply Assumption 3 (b) are given in [6] Proposition 3 and Remark 3.

(b) The differential equations studied in this paper are of more special form than those studied in [6], but the order of the differential equation can be of order $2m$ with m integer, while in [6] the differential equation is of second order i.e. $m = 1$ and it is an open question whether the order $2m$ can also be studied. In addition, considering the special class of differential equations studied in this paper, the present Theorem 2 (for the case $m = 1$) is more general than the corresponding theorem in [6].

References

[1] F.E. BROWDER: Existence theorems for nonlinear partial

differential equations, "Global Analysis", Proc. Symp.Pure Math.,Vol.XVI(held at the University of California,Berkeley,July 1-26,1968),Amer. Math.Soc.Providence,Rhode Island 1970.

[2] F.E. BROWDER, BUI AN TON: Nonlinear functional equations in Banach spaces and elliptic superregularization;Math.Zeitschr.105(1968),177-195.

[3] BUI AN TON: Pseudo-monotone operators in Banach spaces and nonlinear elliptic equations, Math.Zeitschr. 121(1971),243-252.

[4] P. HESS: A strongly nonlinear elliptic boundary value problem, J.Math.Anal.Appl.43(1973),241-249.

[5] A. KRATOCHVÍL: Les méthodes approximatives de la solution des équations elliptiques non linéaires, Comment.Math.Univ.Carolinae 9(1968),455-510.

[6] W. PETRY: Existence theorems for operator equations and nonlinear elliptic boundary-value problems, Comment.Math.Univ.Carolinae 14(1973),27-46. Correction: Comment.Math.Univ.Carolinae 15(1974), 377-378.

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