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COMPOSITION OF PRERADICALS

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Praha

Abstract: This paper provides several constructions of preradicals, namely intersections and sums of preradicals and two types of composition of preradicals. The results which are of technical character, are useful in further development of the theory of preradicals.

Key words: Preradical, intersection of a given family of preradicals, sum of a given family of preradicals.

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In this paper, several constructions of preradicals are provided. These constructions are of technical character, nevertheless they have many important applications in other branches of the theory of preradicals. In the following, we use the terminology and results of [1],[2] and [3] without stating it explicitly.

In what follows R stands for an associative ring with identity and R-mod denotes the category of all unitary left R-modules. The injective hull of a module M will be denoted by $E(M)$, the direct product (direct sum) by $\prod_{i\in I} M_i$ ($\coprod_{i\in I} M_i, M_1 \oplus M_2$). A preradical r

for R-mod is any subfunctor of the identity functor, i.e. $\varkappa$ assigns to each module M its submodule $\varkappa(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $\varkappa(M)$ into $\varkappa(N)$ by restriction. We shall denote by $\mathcal{T}_\varkappa$ ($\mathcal{F}_\varkappa$) the class of all modules M such that $\varkappa(M) = M$ ($\varkappa(M) = 0$) . A preradical $\varkappa$ is said to be

- idempotent if $\varkappa(\varkappa(M)) = \varkappa(M)$ for every module M ,
- a radical if $\varkappa(M/\varkappa(M)) = 0$ for every module M ,
- hereditary if $\varkappa(N) = N \cap \varkappa(M)$ for every submodule N of a module M ,
- cohereditary if $\varkappa(M/N) = (\varkappa(M)+N)/N$ for every submodule N of a module M ,
- splitting if every module splits (a module M splits if $\varkappa(M)$ is a direct summand of M),
- stable if every injective module splits,
- costable if every projective module splits,
- cosplitting if it is both hereditary and cohereditary.

There are several preradicals associated with every preradical $\varkappa$. The idempotent core $\bar{\varkappa}$ is defined by $\bar{\varkappa}(M) = \sum K$, where K runs through all the submodules K of M with $K \in \mathcal{T}_\varkappa$, and the radical closure $\tilde{\varkappa}$ is defined by $\tilde{\varkappa}(M) = \bigcap L$, where L runs through all the submodules L of M such that $M/L \in \mathcal{F}_\varkappa$. Further, the hereditary closure $h(\varkappa)$ is defined by $h(\varkappa)(M) = M \cap \varkappa(E(M))$ and the cohereditary core $ch(\varkappa)$ by $ch(\varkappa)(M) = \varkappa(R)M$. Finally, if $\varkappa$, σ are preradicals then we shall say that $\varkappa \subseteq \sigma$ if $\varkappa(M) \subseteq \sigma(M)$ for all $M \in$ R-mod .

<u>Proposition 1</u>. Let κ_i, $i \in I$ be a family of preradicals and $\kappa(M) = \bigcap_{i \in I} \kappa_i(M)$ for all $M \in R\text{-}mod$. Then

(i) κ is a preradical,

(ii) $\mathcal{T}_\kappa = \bigcap_{i \in I} \mathcal{T}_{\kappa_i}$ and $\mathcal{F}_{\kappa_i} \subseteq \mathcal{F}_\kappa$ for all $i \in I$,

(iii) κ is a radical provided each κ_i is so,

(iv) κ is hereditary provided each κ_i is so,

(v) $h(\kappa) = \bigcap_{i \in I} h(\kappa_i)$,

(vi) κ is stable provided each κ_i is so,

(vii) if R is left perfect and each κ_i is cosplitting then κ is cosplitting.

<u>Proof</u>. (i) For each $M \in R\text{-}mod$ there is a set $I_M \subseteq I$ such that for all $i \in I$ there exists $j \in I_M$ with $\kappa_j(M) = \kappa_i(M)$. If $f: M \longrightarrow N$ is a homomorphism and $i \in I_M \cup I_N$ then $\kappa(M) \subseteq \kappa_i(M)$, and hence $f(\kappa(M)) \subseteq \kappa_i(N)$. But $\kappa(N) = \bigcap_{i \in I} \kappa_i(N)$ and we see that κ is a preradical.

(ii) It is clear.

(iii) Let $M \in R\text{-}mod$ and $L = I_M \cup I_{M/\kappa(M)}$. Since $\kappa(M) = \bigcap_{i \in L} \kappa_i(M)$, there is a monomorphism $f: M/\kappa(M) \to$ $\longrightarrow \prod_{i \in L} M/\kappa_i(M)$.

(iv) Let $M \in R\text{-}mod$, $N \subseteq M$ be a submodule and $L = I_M \cup I_N$. Then $\kappa(N) = \bigcap_{i \in I} \kappa_i(N) = \bigcap_{i \in I} (\kappa_i(M) \cap N) =$ $= N \cap (\bigcap_{i \in I} \kappa_i(M)) = N \cap \kappa(M)$.

(v) For every injective module Q, $\bigcap_{i \in I} h(\kappa_i)(Q) =$ $= \bigcap_{i \in I} \kappa_i(Q) = \kappa(Q) = h(\kappa)(Q)$. However,

$\bigcap_{i\in I} h(r_i)$ is hereditary and we are through by [2], 2.7.

(vi) With respect to [3], 2.6 we may assume that all r_i, $i \in I$, are hereditary. Now it suffices to use (iv), (ii) and [3], 2.4.

(vii) Let $0 \longrightarrow A \longrightarrow P \longrightarrow T \longrightarrow 0$ be a projective cover of a module $T \in \mathcal{T}_r$. By [2], 4.2, $P \in \mathcal{T}_{r_i}$, for all $i \in I$, and therefore $P \in \mathcal{T}_t$. An application of [2], 4.3 and (iv) completes the proof.

Corollary 2. Let r be a preradical. Then

(i) $\tilde{r} = \bigcap s$, where s runs through all the radicals containing r,

(ii) $h(r) = \bigcap t$, where t runs through all the hereditary preradicals containing r,

(iii) $h(r) = \bigcap u$, where u runs through all the hereditary radicals containing r.

Proposition 3. Let r, s be preradicals and $(r \circ s)(M) = r(s(M))$ for all $M \in R\text{-}mod$. Then

(i) $r \circ s$ is a preradical,

(ii) $\mathcal{T}_{r\circ s} = \mathcal{T}_r \cap \mathcal{T}_s$ and $\mathcal{F}_r \cup \mathcal{F}_s \subseteq \mathcal{F}_{r\circ s}$,

(iii) $\overline{r \cap s} \subseteq r \circ s \subseteq r \cap s$,

(iv) if $r \cap s$ is idempotent then $r \circ s = s \circ r =$
$= r \cap s$.

Proof. (i) and (ii) are obvious and (iv) is an immediate consequence of (iii).

(iii) The inclusion $r \circ s \subseteq r \cap s$ is trivial. By (ii), Prop.1 (ii) and [1], $\mathcal{T}_{\kappa\circ\sigma} = \mathcal{T}_{\overline{\kappa\cap\sigma}}$. Hence for all $M \in R\text{-}mod$, $\overline{\kappa\cap\sigma}(M) \in \mathcal{T}_{\kappa\circ\sigma}$ and consequently $\overline{\kappa\cap\sigma}(M) \subseteq (\kappa\circ\sigma)(M)$.

Proposition 4. Let κ, σ be proradicals. Then

(i) if κ is hereditary then $\kappa\circ\sigma = \kappa\cap\sigma$,

(ii) if σ is hereditary and κ is idempotent then $\kappa\circ\sigma$ is idempotent and $\kappa\circ\sigma = \overline{\kappa\cap\sigma}$,

(iii) if both κ and σ are hereditary then $\kappa\circ\sigma = \sigma\circ\kappa = \kappa\cap\sigma$ is hereditary,

(iv) if σ is hereditary then $\overline{\kappa\circ\sigma} = \overline{\kappa}\circ\overline{\sigma} = \overline{\kappa}\circ\sigma$,

(v) if both κ and σ are stable (costable, splitting) then $\kappa\circ\sigma$ is so.

Proof. (i), (iii) and (v) are obvious.

(ii) For all $M \in R\text{-}Mod$, $(\kappa\circ\sigma)(M) = \kappa(\sigma(M)) \in \mathcal{T}_\sigma$ and consequently $\kappa(\sigma(\kappa(\sigma(M)))) = \kappa(\kappa(\sigma(M))) = \kappa(\sigma(M))$.

(iv) By (ii), $\overline{\kappa}\circ\overline{\sigma}$ is idempotent. Further, $\mathcal{T}_{\overline{\kappa}\circ\overline{\sigma}} = \mathcal{T}_{\overline{\kappa}} \cap \mathcal{T}_{\overline{\sigma}} = \mathcal{T}_\kappa \cap \mathcal{T}_\sigma = \mathcal{T}_{\kappa\circ\sigma} = \mathcal{T}_{\overline{\kappa\circ\sigma}}$ and we are through by [1].

Proposition 5. Let κ, σ be preradicals for $R\text{-}Mod$. Then

(i) if both κ and σ are radicals then $\kappa\circ\sigma$ is a radical,

(ii) if σ is a radical then $\widetilde{\kappa\circ\sigma} = \widetilde{\kappa}\circ\widetilde{\sigma} = \widetilde{\kappa}\circ\sigma$,

(iii) if both κ and σ are cohereditary then $\kappa\circ\sigma$ is so,

(iv) if κ is cosplitting and σ is cohereditary then $\kappa\circ\sigma = ch(\kappa\cap\sigma)$,

(v) if both r and s are cohereditary and s is costable then $r \circ s = ch(r \cap s)$,

(vi) if both r and s are cohereditary and costable then $r \circ s = s \circ r$,

(vii) if both r and s are cosplitting then $r \circ s = s \circ r$ is cosplitting,

(viii) if R is commutative and both r and s are cohereditary then $r \circ s = s \circ r$.

Proof. (i) According to [1], $r(s(M/r(s(M)))) = r(s(M)/r(s(M))) = 0$.

(ii) By (i), $\widetilde{r \circ s} \subseteq \tilde{r} \circ \tilde{s}$. Let $M \in R\text{-Mod}$ and $N = \widetilde{r \circ s}(M)$. Then $\widetilde{r \circ s}(M/N) = 0$, i.e. $0 = r(s(M/N)) = (\tilde{r} \circ \tilde{s})(M/N)$. Thus $(\tilde{r} \circ \tilde{s})(M) \subseteq N$.

(iii) By [2], 4.8, $r(s(M)) = r(s(R)M) = r(R)s(R)M$ and [2], 4.10 yields the desired result.

(iv) With respect to [2], 4.8, it is sufficient to show that $(r \circ s)(R) = r(R)s(R) = ch(r \cap s)(R) = r(R) \cap s(R)$. However, this equality is an easy consequence of the fact that $r(R)$ satisfies the condition (a) (see [2], 4.8).

(v) Since $s(R)$ is a direct summand of R as a left ideal, $r(R)s(R) = r(R) \cap s(R)$.

(vi) It follows immediately from (v).

(vii) By (iv), (iii) and Prop. 4 (iii).

(viii) In view of [2], 4.8 it is enough to observe that $(r \circ s)(R) = r(R)s(R) = s(R)r(R) = (s \circ r)(R)$.

Proposition 6. Let r, s be preradicals for $R\text{-Mod}$. Then

(i) if either r is hereditary or s is stable then $h(r \circ s) = h(r) \circ h(s)$,

(ii) if s is stable then $\tilde{h}(r \circ s) = \tilde{h}(r) \circ \tilde{h}(s)$.

Proof. (i) It suffices to prove that $h(r \circ s)(Q) = (h(r) \circ h(s))(Q)$ for every injective module Q (see [2], 2.7). But $h(r \circ s)(Q) = (r \circ s)(Q) = r(s(Q))$ and $(h(r) \circ h(s))(Q) = h(r)(h(s)(Q)) = h(r)(s(Q))$. If r is hereditary then $r(s(Q)) = h(r)(s(Q))$. If s is stable then $s(Q)$ is injective and consequently $h(r)(s(Q)) = r(s(Q))$.

(ii) By (i), Prop.5 (ii) and [3], 2.6.

Proposition 7. Let r, s be preradicals. Then

(i) if r is cohereditary then $ch(r \circ s) = ch(r) \circ ch(s) = r \circ ch(s)$,

(ii) if $s(R)$ is projective (in particular, if s is costable) then $ch(r \circ s) = ch(r) \circ ch(s)$.

Proof. (i) Obvious, since $r(s(R)) = r(R)s(R)$.

(ii) We have $ch(r \circ s)(R) = (r \circ s)(R) = r(s(R))$ and $(ch(r) \circ ch(s))(R) = r(R)s(R)$. However, $s(R)$ is projective and we are through by [1].

Proposition 8. Let $r_1, r_2, \dots, r_n$ be preradicals and $r = \bigcap_{i=1}^{n} r_i$. Then r is costable (cosplitting, splitting hereditary) provided that each r_i is so.

Proof. Let r, s be two costable preradicals. Since $ch(r \cap s)(R) = ch(ch(r) \cap ch(s))$, $ch(r \cap s) = ch(ch(r) \cap ch(s))$.

By [3], 3.8 and Prop.5 (v), $ch(ch(\varkappa)\cap ch(s)) = ch(\varkappa)\circ ch(s)$. Hence $ch(\varkappa\cap s)$ is costable by Prop.4 (v) and we are through due to [3], 3.8. The rest follows from Prop. 4.

Proposition 9. Let $\varkappa$ be a preradical, $\varkappa^1 = \varkappa$, $\varkappa^{\alpha+1} = \varkappa\circ\varkappa^{\alpha}$ for every ordinal $\alpha \geq 1$ and $\varkappa^{\alpha} = \bigcap_{\beta<\alpha}\varkappa^{\beta}$, for α being a limit ordinal. Then $\bar{\varkappa} = \bigcap_{\alpha}\varkappa^{\alpha}$.

Proof. Let $s = \bigcap_{\alpha}\varkappa^{\alpha}$. Since $\bar{\varkappa}\circ\bar{\varkappa} = \bar{\varkappa}$ and $\bar{\varkappa}\subseteq\varkappa$, $\bar{\varkappa}\subseteq s$. On the other hand, if $M\in R\text{-}Mod$, then there exists an ordinal α such that $s(M) = \varkappa^{\alpha}(M)$. We have $\varkappa(s(M)) = (\varkappa\circ s)(M) = \varkappa^{\alpha+1}(M) = \varkappa^{\alpha}(M) = s(M)$. Thus $s(M)\subseteq\bar{\varkappa}(M)$.

The proofs of Propositions 10 - 18 are dual to that of Propositions 1 - 9 and therefore will be omitted.

Proposition 10. Let $\varkappa_i$, $i\in I$ be a family of preradicals and $\varkappa(M) = \sum_{i\in I}\varkappa_i(M)$ for all $M\in R\text{-}Mod$. Then

(i) $\varkappa$ is a preradical,

(ii) $\mathcal{F}_{\varkappa} = \bigcap_{i\in I}\mathcal{F}_{\varkappa_i}$ and $\mathcal{T}_{\varkappa_i}\subseteq\mathcal{T}_{\varkappa}$ for all $i\in I$,

(iii) $\varkappa$ is idempotent provided each $\varkappa_i$ is so,

(iv) $\varkappa$ is cohereditary provided each $\varkappa_i$ is so,

(v) $ch(\varkappa) = \sum_{i\in I} ch(\varkappa_i)$,

(vi) if R is left perfect then $\varkappa$ is costable provided each $\varkappa_i$ is so,

(vii) $\varkappa$ is cosplitting provided each $\varkappa_i$ is so.

Corollary 11. Let $\varkappa$ be a preradical. Then

(i) $\bar{\varkappa} = \sum s$, where s runs through all the idempotent preradicals contained in r ,

(ii) $ch(r) = \Sigma t$, where t runs through all the cohereditary radicals contained in r .

Proposition 12. Let r, s be two preradicals. For every $M \in R\text{-}Mod$ let $(r \Delta s)(M) = X$, where $X/r(M) = s(M/r(M))$. Then

(i) $r \Delta s$ is a preradical,

(ii) $\mathcal{F}_{r\Delta s} = \mathcal{F}_r \cap \mathcal{F}_s$ and $\mathcal{T}_r \cup \mathcal{T}_s \subseteq \mathcal{T}_{r\Delta s}$,

(iii) $r + s \subseteq r \Delta s \subseteq \widetilde{r+s}$,

(iv) if $r+s$ is a radical then $r+s = r \Delta s = s \Delta r$.

Proposition 13. Let r, s be preradicals. Then

(i) if s is cohereditary then $r \Delta s = r + s$,

(ii) if r is cohereditary and s is a radical then $r \Delta s$ is a radical and $r \Delta s = \widetilde{r+s}$,

(iii) if both r and s are cohereditary then $r \Delta s = r+s = s \Delta r$ is cohereditary,

(iv) if r is cohereditary then $\widetilde{r \Delta s} = \tilde{r} \Delta \tilde{s} = r \Delta \tilde{s}$,

(v) if both r and s are stable (costable, splitting) then $r \Delta s$ is so.

Proposition 14. Let r, s be preradicals. Then

(i) if both r and s are idempotent then $r \Delta s$ is idempotent,

(ii) if r is idempotent then $\overline{r \Delta s} = \bar{r} \Delta \bar{s} = r \Delta \bar{s}$,

(iii) if both r and s are hereditary then $r \Delta s$ is hereditary,

(iv) if both r and s are hereditary and r is stable then $r \Delta s = h(r+s)$,

(v) if both r and s are stable hereditary then $r \Delta s = s \Delta r$,

(vi) if both r and s are cosplitting then $r \Delta s = s \Delta r$ is cosplitting.

Proposition 15. Let r, s be preradicals for R-*Mod*. Then

(i) if either r is costable or s is cohereditary then $ch(r \Delta s) = ch(r) \Delta ch(s)$,

(ii) if r is costable then $\overline{ch}(r \Delta s) = \overline{ch}(r) \Delta \overline{ch}(s)$.

Proposition 16. Let r, s be preradicals for R-*Mod*. Then

(i) if s is hereditary then $h(r \Delta s) = h(r) \Delta h(s) = h(r) \Delta s$,

(ii) if either r is stable or R is left hereditary then $h(r \Delta s) = h(r) \Delta h(s)$.

Proposition 17. Let $r_1, r_2, \dots, r_m$ be preradicals and $r = \sum_{i=1}^{m} r_i$. Then r is stable (costable, splitting cohereditary) provided each r_i is so.

Proposition 18. Let r be a preradical, $r_1 = r$, $r_{\alpha+1} = r_\alpha \Delta r$ for every ordinal $\alpha \geq 1$ and $r_\alpha = \sum_{\beta < \alpha} r_\beta$, for α being a limit ordinal. Then $\tilde{r} = \sum_\alpha r_\alpha$.

Proposition 19. Let r, s be two preradicals and t_i , $i \in I$, be a family of preradicals. Then

$(\sum_{i\in I} t_i)\circ \varkappa = \sum_{i\in I}(t_i\circ\varkappa)$, $(\bigcap_{i\in I} t_i)\circ\varkappa = \bigcap_{i\in I}(t_i\circ\varkappa)$,

$\sigma\,\Delta \sum_{i\in I} t_i = \sum_{i\in I}(\sigma\,\Delta\, t_i)$ and $\sigma\,\Delta\,(\bigcap_{i\in I} t_i) =$
$= \bigcap_{i\in I}(\sigma\,\Delta\, t_i)$. Moreover, if $\varkappa$ is hereditary and σ is cohereditary then $\varkappa\circ\sum_{i\in I} t_i = \sum_{i\in I}(\varkappa\circ t_i)$, $\varkappa\circ\bigcap_{i\in I} t_i =$
$= \bigcap_{i\in I}(\varkappa\circ t_i)$, $(\sum_{i\in I} t_i)\,\Delta\,\sigma = \sum_{i\in I}(t_i\,\Delta\,\sigma)$ and $(\bigcap_{i\in I} t_i)\,\Delta\,\sigma = \bigcap_{i\in I}(t_i\,\Delta\,\sigma)$.

Proof. It is rather of technical character and runs without difficulties.

Theorem 20. Let $\varkappa, \sigma$ be preradicals. Then $\widetilde{\varkappa\circ\sigma} = \tilde{\varkappa}\circ\tilde{\sigma}$ provided that at least one of the following conditions holds:

(i) σ is a radical,

(ii) $\varkappa$ is hereditary and $\widetilde{\varkappa\circ\sigma}$ is cohereditary,

(iii) $\varkappa$ is idempotent cohereditary and $\widetilde{\varkappa\circ\sigma}$ is cohereditary.

Proof. (i) By Prop. 5 (ii).

(ii) and (iii). The inclusion $\widetilde{\varkappa\circ\sigma} \subseteq \tilde{\varkappa}\circ\tilde{\sigma}$ is obvious. To prove the inverse inclusion, it is sufficient to show that $\tilde{\varkappa}(\tilde{\sigma}(M)) = 0$ for all $M \in \mathcal{F}_{\varkappa\circ\sigma}$. Let $\alpha \geq 2$ be an ordinal and let $\varkappa(\sigma_\beta(M)) = 0$ for all $\beta < \alpha$. If $\alpha - 1$ exists then we have the exact sequence $0 \longrightarrow \sigma_{\alpha-1}(M) \longrightarrow \sigma_\alpha(M) \longrightarrow \sigma(M/\sigma_{\alpha-1}(M)) \longrightarrow 0$. By the induction hypothesis, $\varkappa(\sigma_{\alpha-1}(M)) = 0$. Further, $\widetilde{\varkappa\circ\sigma}$ is cohereditary and $(\varkappa\circ\sigma)(M) = \varkappa(\sigma(M)) = 0$ (since $M \in \mathcal{F}_{\varkappa\circ\sigma}$). Hence $\varkappa(\sigma(M/\sigma_{\alpha-1}(M))) = 0$, and consequently $\varkappa(\sigma_\alpha(M)) = 0$, since $\varkappa$ is idempotent. If α is a limit ordinal then $\sigma_\alpha(M) = \bigcup_{\beta<\alpha} \sigma_\beta(M)$. If $\varkappa$ is hereditary then $0 = r(s_\beta(M)) = s_\beta(M) \cap r(s_\alpha(M))$ for all $\beta < \alpha$, and consequently $r(s_\alpha(M)) = 0$. Now, let r be cohereditary. Then $r(s_\alpha(M)) = 0$, since there exists an epimorphism f of the r-torsionfree module $\coprod_{\beta<\alpha} s_\beta(M)$ onto $s_\alpha(M)$. We have proved that $r(s_\alpha(M)) = 0$ for every ordinal α, and therefore $r(\tilde{s}(M)) = 0$. Thus $\tilde{r}(\tilde{s}(M)) = 0$ (since $\mathcal{F}_r = \mathcal{F}_{\tilde{r}}$) and we are through.

Corollary 21. Let r be a hereditary preradical and s be a preradical such that $\widetilde{r \cap s}$ is a cohereditary radical. Then $\widetilde{r \cap s} = \tilde{r} \cap \tilde{s}$.

Proof. By Prop. 4 (i), Th. 20 and [2], 2.3.

Corollary 22. Let r_i, $i \in I$ be a family of hereditary preradicals such that $r_i \cap r_j = zer$ (where zer is the zero functor) whenever $i \neq j$. Let $s = \sum_{i \in I} r_i$ and $T \in \mathcal{T}_s$. Then T is a direct sum of its submodules $\tilde{r}_i(T)$, $i \in I$.

Proof. It suffices to show that $\tilde{r}_i \cap \sum_{j \neq i} \tilde{r}_j = zer$ for all $i \in I$. However, $\tilde{r}_i \cap \sum_{j\neq i} \tilde{r}_j = \sum_{j \neq i} (\tilde{r}_i \cap \tilde{r}_j) = \sum_{j\neq i} \widetilde{r_i \cap r_j} = zer$, using Cor. 21, Prop.19 and Prop. 4 (i).

The proof of the following proposition is left to the reader as an easy exercise.

Proposition 23. Let r, s, t be preradicals. Then

(i) $(r \circ s) \circ t = r \circ (s \circ t)$,

(ii) $(r \,\Delta\, s)\,\Delta\, t = r \,\Delta\,(s \,\Delta\, t)$,

(iii) $(r \,\Delta\, s) \circ t = [(r \circ t)\,\Delta\, s] \circ t$ provided t is idempotent,

(iv) $(r \,\Delta\, s) \circ t = (r \circ t) \,\Delta\, (s \circ t)$ provided t is a radical,

(v) $(r \circ s) \,\Delta\, r = r = r \,\Delta\, (s \circ r)$ provided r is a radical,

(vi) $r \circ (r \,\Delta\, s) = r = (s \,\Delta\, r) \circ r$ provided r is idempotent.

Example 24. Consider the following three preradicals for the category of Abelian groups: $r(G)$ is the maximal divisible subgroup of G, $s(G)$ is a 2-socle of G and $t(G) = 2G$. Then r is an idempotent radical, s is a hereditary preradical, t is a cohereditary radical, $\mathit{ker} = \widetilde{r \circ s} \neq \tilde{r} \circ \tilde{s} = \tilde{s}$ and $\mathit{ker} = \widetilde{t \circ s} \neq \tilde{t} \circ \tilde{s}$. Thus the hypotheses of Theorem 20 cannot be weakened.

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