Commentationes Mathematicae Universitatis Carolinae

Jiří Veselý

Some properties of a generalized heat potential (Preliminary communication)

Commentationes Mathematicae Universitatis Carolinae, Vol. 15 (1974), No. 2, 357--360

Persistent URL: http://dml.cz/dmlcz/105560

Commentationes Mathematicae Universitatis Carolinae

15,2 (1974)

SOME PROPERTIES OF A GENERALIZED HEAT POTENTIAL

(Preliminary communication)

Jiří VESELÝ, Praha

Abstract: A generalized heat potential and its continuous extension from an open set with non-smooth boundary to its closure is studied.

Key words: generalized heat potential, boundary behaviour

AMS: 31B10 Ref.Ž.: 7.972.26

For $x = [x_1, \ldots, x_{m+1}] \in R^{m+1}$, $m \geq 3$ we shall write $x = [\hat{x}, x_{m+1}] = [x, t]$ where $x \in R^m$, $t \in R^1$. Similarly for the differential operator $\nabla = [\partial_1, \ldots, \partial_{m+1}]$ we put $\hat{\nabla} = [\partial_1, \ldots, \partial_m]$. Let G be the function defined on R^{m+1} by

$$G(x) = x_{m+1}^{-\frac{m}{2}} \cdot exp(-\|\hat{x}\| / 4x_{m+1}) \quad \text{for } x_{m+1} > 0,$$

$$G(x) = 0 \quad \text{for } x_{m+1} \leq 0.$$

Suppose D is an open set in R^{m+1} with the boundary B for which $B_\tau = B \cap \{[x, t] \in R^{m+1}, t \leq \tau\}$ is compact for any $\tau \in R^1$.

$\mathcal{C}$ will denote the collection of all bounded continuous functions on B and $\mathcal{D}$ will be the space of all infinitely differentiable functions φ with compact support $spt\ \varphi \in R^{m+1}$.

For any $x \in R^{m+1}$ and $\varphi \in \mathcal{D}(x) = \{\varphi \in \mathcal{D};\ x \notin spt\ \varphi\}$ we define

$$T\varphi(x) = -\int_D (\hat{\nabla}_w G(x-w) \cdot \hat{\nabla}\varphi(w) + G(x-w)\partial_{m+1}\varphi(w))\,dw \,.$$

The integral on the right-hand side is finite for any $\varphi \in \mathcal{D}(x)$. As $T\varphi(x)$ depends on values of φ in a neighborhood of boundary B only we can define $T\varphi(x)$ even for any $\varphi \in \mathcal{D}$ by means of

$$T\varphi(x) \stackrel{def}{=} T\tilde{\varphi}(x)$$

where $\tilde{\varphi} \in \mathcal{D}(x)$ and $\varphi(x) = \tilde{\varphi}(x)$ in a neighborhood of B. $T\varphi(x)$ may be considered as a distribution over $\mathcal{D}$ and it is closely connected with classical heat potentials of single and double layer.

Three following questions are solved:

(1) When there is a measure ν_x such that

$$T\varphi(x) = \int \varphi\, d\nu_x = \langle \varphi, \nu_x \rangle$$

for every $\varphi \in \mathcal{D}(x)$?

Replacing φ by f we can define $Tf(x) = \langle f, \nu_x \rangle$ for any $f \in \mathcal{C}$ provided ν_x from (1) exists.

(2) When $Tf(x)$ is a well-defined function of the variable x on D for any $f \in \mathcal{C}$?

(3) When this function Tf defined on D can be continuously extended from D to $D \cup B$ for any $f \in \mathcal{C}$?

The case $m = 1$ was investigated for special D by M. Dont in [1] and similar questions were solved by J. Král in [2],[3] and by the author in [4].

Recall that for a measurable set $M \subset R^{m+1}$ its perimetr $P(M)$ is defined by

$$P(M) = \sup_{\omega} \int_M div\, \omega(w)\, dw$$

where $\omega = [\omega_1, \dots, \omega_{m+1}]$ ranges over system of all functions with components $\omega_i \in \mathcal{D}$, $i = 1, 2, \dots, m+1$ satisfying

$$\sum_{j=1}^{m+1} \omega_j^2(w) \le 1, \quad w \in R^{m+1}$$

Put $\Gamma = \{x \in R^m;\ \|x\| = 1\}$, $Z = (0, \infty) \times \Gamma$. We define for any $z = [x, t]$ and $(\varphi, \gamma, \theta) \in (0, \infty) \times (0, \infty) \times \Gamma$

$$S_z(\varphi, \gamma, \theta) = [\hat{z} + \varphi\theta, z_{m+1} - \frac{\varphi^2}{4\gamma}] .$$

Given $(\gamma, \theta) \in Z$ let $S(\gamma, \theta)$ be the parabola described by $S_z(\cdot, \gamma, \theta)$ on $(0, \infty)$. A point $b \in S = S(\gamma, \theta)$ is termed a hit of the parabola S on D provided each neighborhood of b meets both $S \cap D$ and $S - D$ in a set of positive H_1-measure where H_k is the k-dimensional Hausdorff measure. The number of all hits of $S(\gamma, \theta)$ on D will be denoted by $n(z, \gamma, \theta)$. We put for any $z \in R^{m+1}$

$$v(z) = \int_Z e^{-\gamma} \gamma^{\frac{m}{2}-1} n(z, \gamma, \theta)\, dH_m((\gamma, \theta)) .$$

The function v which is called the parabolic variation of D is a lower semicontinuous function on R^{m+1}.

The answers to questions (1) - (3) can be formulated now in a form of necessary and sufficient conditions corresponding to (1) - (3) as follows:

(1) $v(x) < \infty$,

(2) $P(D_\tau) < \infty$ for all $\tau \in R^1$ where

$D_\tau = D \cap \{[x,t] \in R^{m+1}; t < \tau\}$,

(3) $\sup\{v(\varsigma); \varsigma \in B_\tau\} < \infty$ for all $\tau \in R^1$.

Complete proofs of the formulated results and some further details are contained in a paper submitted for the publication in Czechoslovak Mathematical Journal.

References

[1] M. DONT: On a planar heat potential (to appear in Czech. Math.Journal).

[2] J. KRÁL: The Fredholm method in potential theory, Trans, Amer.Math.Soc. 125(1966),511-547.

[3] J KRÁL: Flows of heat and the Fourier problem, Czech. Math.Journal 20(1970),556-598.

[4] J. VESELÝ: On the heat potential of the double distribution, Čas.pro pěst.mat.98(1973),181-198.

Matematicko-fyzikální fakulta
Karlova universita
Malostranské nám.25
110 00 Praha 1

(Oblatum 22.4.1974)