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Ladislav Beran

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MODULARITY IN GENERALIZED ORTHOMODULAR LATTICES

Ladislav BERAN, Praha

Abstract: The purpose of this paper is to characterize the generalized orthomodular lattices which are solvable in the class of modular lattices.

Key words: Orthomodular lattice, commutator, solvability, modularity.

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1. Preliminaries. Recall that a lattice $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ is said to be orthomodular iff it satisfies the following conditions:

(i) $a \vee a' = 1$;

(ii) $a \leqq b \Longrightarrow a' \geqq b'$;

(iii) $(a')' = a$;

(iv) $s \geqq t' \ \& \ s \wedge t = 0 \Longrightarrow s = t'$.

The element a' is called an orthocomplement of a .

By a generalized orthomodular lattice $\mathcal{G}$, one means a lattice $\mathcal{G} = (G, \vee, \wedge, 0)$ such that

(j) for every $a \neq 0$, $a \in G$, the interval $[0, a]$ determines an orthomodular lattice

$\mathcal{G}(0, a) = ([0, a], \vee, \wedge, a - x, 0, a)$;

(jj) for $x \leqq a \leqq b$ of G and for the orthocomplements $a-x$, $b-x$ of x in $[0,a]$ and of x in $[0,b]$, respectively,

$$a-x = (b-x) \wedge a \quad .$$

Basic facts on orthomodular lattices are used here without comment; a relatively complete background may be obtained from [2].

Let $\mathcal{G}$ denote a generalized orthomodular lattice, let $\mathcal{G}^{(0)} = \mathcal{G}$ and let $\mathcal{G}^{(n)}$ $(n \geqq 1)$ be the ideal of $\mathcal{G}$ generated by all the commutators $com_{[0,a]}(x,y) = (x \vee y) \wedge \wedge (x \vee (a-y)) \wedge ((a-x) \vee y) \wedge ((a-x) \vee (a-y))$ where $x, y \leqq a \in G^{(n-1)}$. We shall call $\mathcal{G}^{(n)}$ the <u>n-th commutator sublattice</u> of $\mathcal{G}$.

We remark that from (jj) above it is clear that in the definition of $\mathcal{G}^{(n)}$ we can demand $x, y \in G^{(n-1)}$, $a \in G$. It is easy to see that $\mathcal{G}^{(n)}$ is a generalized orthomodular lattice for every $n \geqq 0$.

A generalized orthomodular lattice $\mathcal{G}$ is said to be <u>solvable in a class</u> $\mathcal{C}$ <u>of lattices</u> if there exists $n \in \mathbb{N}$ such that $\mathcal{G}^{(n)}$ belongs to $\mathcal{C}$. It is known that a lattice $\mathcal{G}$ is solvable in the class $\mathcal{D}$ of distributive lattices iff it is distributive; see for example [1]. On the other hand, it is easily seen that a lattice $\mathcal{G}$ is solvable in $\mathcal{D}$ iff it is solvable in the sense of Marsden [3].

2. <u>Solvability in the class</u> $\mathcal{M}$. In this section we shall prove a characterization of those generalized orthomodular lattices which are solvable in the class $\mathcal{M}$ of

modular lattices. The technique of the proof has one remarkable feature: Only some elementary facts on orthomodular lattices are used and the key construction of the pentagon determined by $0, A, B, C, c_3$ has a nice geometric interpretation as a translation in the lattice $\mathcal{G}$.

Theorem. A generalized orthomodular lattice $\mathcal{G}$ is solvable in the class $\mathcal{M}$ iff it is modular.

Proof. 1) If $\mathcal{G}$ is modular, then $\mathcal{G}'$ is modular, too.

2) Let $\mathcal{G}^{(n)}$ be modular, $n \geqq 1$. If $\mathcal{G}^{(n-1)}$ is not modular, then there exists a five-element nonmodular lattice $\mathcal{N}_5$ determined by elements $\omega < \alpha < \gamma < \iota, \beta$. Let $b = \iota - \beta$, $c = \iota - \alpha$, $a = \iota - \gamma$, $i = \iota - \omega$. The elements $0 < a < c < i$, b define a sublattice isomorphic to $\mathcal{N}_5$. Since $i \leqq \iota$, it is a sublattice of $\mathcal{G}^{(n-1)}$. For an element $x \in [0, i]$ we shall write $x^+ = i - x$. Now we have

$$c_1 = com_{[0,i]}(a, b) = (a \vee b^+) \wedge (a^+ \vee b)$$

and, similarly,

$$c_2 = com_{[0,i]}(c, b) = (c \vee b^+) \wedge (c^+ \vee b) .$$

Let $c_3 = c_1 \vee c_2$. Then

$$c_3 = \{[(a \vee b^+) \wedge (a^+ \vee b)] \vee (c \vee b^+)\} \wedge$$
$$\wedge \{[(a \vee b^+) \wedge (a^+ \vee b)] \vee (c^+ \vee b)\} ,$$

since the elements $a \vee b^+$, $a^+ \vee b$ commute with the elements $c \vee b^+$, $c^+ \vee b$ and, hence, the last two elements also commute with the element $(a \vee b^+) \wedge (a^+ \vee b)$. Likewise, this implies that

$$[(a \vee b^+) \wedge (a^+ \vee b)] \vee (c \vee b^+) =$$

$$= (a \vee b^+ \vee c) \wedge (a^+ \vee b \vee c \vee b^+) = c \vee b^+ .$$

Similarly,

$$[(a \vee b^+) \wedge (a^+ \vee b)] \vee (c^+ \vee b) = a^+ \vee b .$$

It follows that $c_3 = (c \vee b^+) \wedge (a^+ \vee b) \in G^{(n)}$. Now let $C = c \wedge c_3 = c \wedge (a^+ \vee b)$, $A = a \wedge c_3 = a \wedge (a^+ \vee b)$, $B = b \wedge c_3 = b \wedge (c \vee b^+)$. Note that $A \leqq C$ and that c_3 commutes with a and b. Hence

$$B \vee A = B \vee C = c_3 \ \& \ B \wedge A = B \wedge C = 0 .$$

However,

$$a \wedge C = A , \ a \vee C = c \wedge (a \vee c_3)$$

and since

$$a \vee c_3 = (a \vee c \vee b^+) \wedge (a \vee a^+ \vee b) =$$

$$= a \vee c \vee b^+ ,$$

we have $a \vee C = c$. But this shows that $[a, c]$ and $[A, C]$ are transposes. Since $a \neq c$, we have $A < C$ and from this we conclude that the five elements $0, A, B, C, c_3$ of $\mathcal{G}^{(n)}$ determine a sublattice isomorphic to $\mathcal{N}_5$. This contradicts the modularity of $\mathcal{G}^{(n)}$. Hence, $\mathcal{G}^{(n-1)}$ is modular, and we are done.

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Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, 18600 Praha 8

Československo

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