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UNIONS IN E-M CATEGORIES AND COREFLECTIVE SUBCATEGORIES ^{x)}

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Abstract: The concept of M -unions in categories is defined and discussed and a characterization of coreflective subcategories by means of this concept is given.

Key-words: M -union, M -image, factorization, coreflective subcategory.

AMS: 18A30, 18A40

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1. Introduction. This paper will be concerned with categorial unions in two settings. First, in an E-M category, M -unions will be defined and discussed. It will be shown that the definition of M -unions can be made stronger than the expected definition and that M -unions exist in many E-M categories.

Second, looking at coreflective subcategories, a characterization of M -coreflective subcategories will be obtained with the use of M -unions and M -images.

Categorial unions have never attracted much attention because coproducts are generally a stronger and more basic idea. However, categorial unions are the generaliza-

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tion of a very intuitive concept that appears in many situations. For example, unions in the category of all topological spaces take on a simpler form than do coproducts and are more useful in applications.

The categorical definitions not stated in this paper can be found in Mitchell [4], MacLane [3], or Herrlich and Strecker [1].

2. E-M category. E-M categories arise naturally in all categories where some notion of images is introduced. This is stated categorically in terms of factorizations of morphisms.

Definition 1. Let ξ be a category and let E and M be classes of morphisms which are closed under composition with all isomorphisms. We call ξ an E-M category if and only if:

1) Every morphism in ξ has an E-M factorization. That is, given a morphism $f: A \rightarrow B$, there exist morphisms $e: A \rightarrow C$ and $m: C \rightarrow B$ with $e \in E$ and $m \in M$ such that $me = f$.

2) ξ has the unique E-M diagonal property. That is, given a commutative square $mq = fe$ with $e \in E$ and $m \in M$, there exists a unique morphism q such that $mq = f$ and $qe = g$.

Examples. Any category is an E-M category, where E is the class of all morphisms (all isomorphisms) and M is the class of all isomorphisms (resp. all morphisms).

The categories of all sets, semigroups, monoids, groups, Abelian groups, rings, commutative rings, and compact Hausdorff spaces are E-M categories where E is the class of all surjective morphisms and M is the class of all injective morphisms.

The categories of all topological spaces, Hausdorff spaces, compact spaces, and connected spaces are E-M categories, where E is the class of all dense maps (surjective maps, quotient maps) and M is the class of all closed embeddings (resp. embeddings, injective maps).

The categories of all topological spaces and all Hausdorff spaces are E-M categories, where E is the class of all final maps and M is the class of all bijective maps.

The categories of all topological spaces, compact spaces, and connected spaces are E-M categories, where E is the class of all bijective maps and M is the class of all cofinal maps.

It follows from the definition that, in an E-M category, E-M factorizations are essentially unique. Therefore, given a morphism $g: A \rightarrow B$ in an E-M category, $g_E: A \rightarrow g(A)$ and $g_M: g(A) \rightarrow B$ will denote the essentially unique E-M factorization of g .

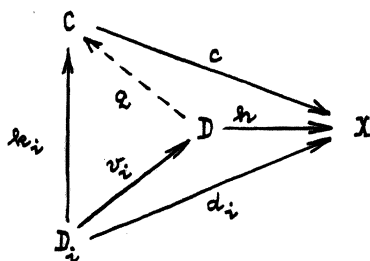
3. M -union. M-unions are a generalization of usual categorical unions.

Definition 2. Let M be a class of morphisms and let $\{d_i: D_i \rightarrow X \mid i \in I\}$ be a family of morphisms in M. Let (D, \mathcal{h}) be a pair, where D is an object and $\mathcal{h}: D \rightarrow X$ is

a morphism in \mathcal{M} such that there exists a family of morphisms $\{v_i: D_i \rightarrow D \mid i \in I\}$ for which $h v_i = d_i$ for all $i \in I$.

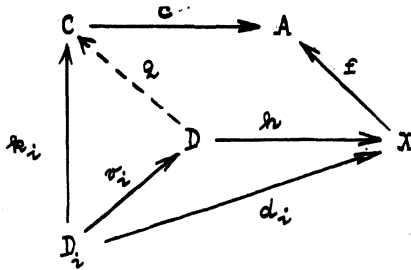
We say (D, h) is the \mathcal{M} -union of $\{d_i \mid i \in I\}$ if and only if

(U) Whenever $c: C \rightarrow X$ is a morphism in \mathcal{M} and $\{h_i: D_i \rightarrow C \mid i \in I\}$ a family of morphisms such that $c h_i = d_i$ for all $i \in I$, it follows that there exists a unique morphism $q: D \rightarrow C$ such that $c q = h$.



We say (D, h) is the strong \mathcal{M} -union of $\{d_i \mid i \in I\}$ if and only if

(SU) Whenever $f: X \rightarrow A$ is a morphism, $c: C \rightarrow A$ a morphism in \mathcal{M} , and $\{h_i: D_i \rightarrow C \mid i \in I\}$ a family of morphisms such that $f d_i = c h_i$ for all $i \in I$, it follows that there exists a unique morphism $q: D \rightarrow C$ such that $c q = f h$.



Strong M -unions are more useful in E-M categories while M -unions suffice in other settings (such as coreflective subcategories):

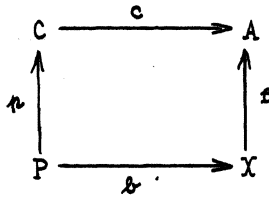
Although the two unions differ by definition, they coincide in E-M categories under very weak hypothesis. More precisely:

Theorem 1. In an E-M category that has weak pullbacks, let $\{d_i: D_i \rightarrow X \mid i \in I\}$ be a family of morphism in M . Let $h: D \rightarrow X$ be a morphism in M through which each d_i factors. Then the following are equivalent:

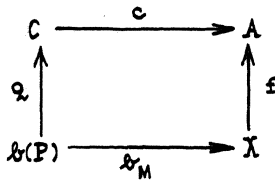
- 1) (D, h) is the strong M -union of $\{d_i \mid i \in I\}$.
- 2) (D, h) is the M -union of $\{d_i \mid i \in I\}$.

Proof. That 1) implies 2) is clear by setting $f = 1_X$ in the definition of strong M -union.

To show 2) implies 1), let $f: X \rightarrow A$ be a morphism $c: C \rightarrow A$ a morphism in M , and $\{k_i: D_i \rightarrow C \mid i \in I\}$ a family of morphisms such that $ck_i = fd_i$ for all $i \in I$. Then let the following diagram be a weak pullback diagram.



By the unique E-M diagonal property, there exists a morphism $q: b(P) \rightarrow C$ such that $cq = fb_M$ and $q\tau = \tau$. Therefore the following is a weak pullback diagram.



Since $cb_i = fd_i$, from the definition of weak pullback there exists for each $i \in I$ a morphism $\alpha_i: D_i \rightarrow b(P)$ such that $q\alpha_i = h_i$ and $b_M\alpha_i = d_i$.

Hence, from the hypothesis, there exists a morphism $\kappa: D \rightarrow b(P)$ such that $b_M\kappa = h$. Therefore $q\kappa: D \rightarrow C$ is a morphism such that $cq\kappa = fb_M\kappa = fh$.

To show uniqueness, let $m, m^*: D \rightarrow C$ be morphisms such that $cm = cm^* = fh$. From the definition of weak pullback, there exist morphisms $d, d^*: D \rightarrow b(P)$ such that $qd = m$, $b_M d = h$, $qd^* = m^*$, and $b_M d^* = h$. But from the hypothesis, $d = d^*$. Therefore $m = qd = qd^* = m^*$.

Examples. In the category of all sets, let \mathcal{M} be the class of all injective functions. Given a family of sets $\{D_i \subseteq X \mid i \in I\}$, the \mathcal{M} -union of their inclusions $d_i: D_i \rightarrow X$ is the pair $(\cup D_i, h)$, where $\cup D_i$ is the usual

set-theoretic union and $\mathcal{h} : \cup D_i \rightarrow X$ is the inclusion function.

In the category of all groups, let \mathcal{M} be the class of all injective homomorphisms. Given a family of subgroups $\{D_i \mid i \in I\}$ of the group X , the \mathcal{M} -union of their inclusion functions $d_i : D_i \rightarrow X$ is the pair $(\langle \{D_i\} \rangle, \mathcal{h})$, where $\langle \{D_i\} \rangle$ is the subgroup generated by the subgroups D_i and $\mathcal{h} : \langle \{D_i\} \rangle \rightarrow X$ is the inclusion homomorphism.

In the category of all topological spaces, let X be a topological space and consider a family of spaces $\{D_i \mid i \in I\}$, where each set D_i is a subset of the set X .

1) When \mathcal{M} is the class of all embeddings and each inclusion $d_i : D_i \rightarrow X$ is an embedding, the \mathcal{M} -union of the d_i is the pair $(\cup D_i, \mathcal{h})$, where $\cup D_i$ is the set-theoretic union of the sets D_i . Here $\cup D_i$ is endowed with the subspace topology and $\mathcal{h} : \cup D_i \rightarrow X$ is the inclusion map.

2) When \mathcal{M} is the class of all injective maps and each inclusion $d_i : D_i \rightarrow X$ is an injective map, the \mathcal{M} -union of the d_i is the pair $(\cup D_i, \mathcal{h})$, where $\cup D_i$ is the set-theoretic union of the D_i . Here $\cup D_i$ is endowed with the topology defined by the following:

A subset O is open in $\cup D_i$ if and only if $O \cap D_i$ is open in D_i for all $i \in I$.

The map $\mathcal{h} : \cup D_i \rightarrow X$ is the inclusion map.

3) Let \mathcal{M} be the class of all closed embeddings and each inclusion $d_i : D_i \rightarrow X$ a closed embedding. Then

the M -union of the d_i is the pair $(cl(\cup D_i), h)$ where $cl(\cup D_i)$ is the closure of the set-theoretic union of the D_i . Here $cl(\cup D_i)$ is endowed with the subspace topology and $h: cl(\cup D_i) \rightarrow X$ is the inclusion map.

For an arbitrary E-M category \mathfrak{E} , it is next shown that the existence of strong M -unions is guaranteed when \mathfrak{E} has coproducts and M consists entirely of monomorphisms.

Proposition 1. In an E-M category where M is a class of monomorphisms, let $\{d_i: D_i \rightarrow X \mid i \in I\}$ be a family of morphisms in M . Let the family of morphisms $\{\mu_i: D_i \rightarrow \coprod D_i \mid i \in I\}$ be the coproduct of the D_i . Furthermore, let $\nu: \coprod D_i \rightarrow X$ be the unique morphism guaranteed by the definition of coproduct such that $\nu\mu_i = d_i$ for all $i \in I$. It then follows that $(\nu(\coprod D_i), \nu_M)$ is the strong M -union of the d_i .

Proof. First, there exists the family of morphisms $\{\nu_E \mu_i: D_i \rightarrow \nu(\coprod D_i) \mid i \in I\}$ such that $\nu_M \nu_E \mu_i = \nu\mu_i = d_i$ for all $i \in I$.

Second, let $f: X \rightarrow A$ be a morphism, $c: C \rightarrow A$ a morphism in M , and $\{h_i: D_i \rightarrow C \mid i \in I\}$ a family of morphisms such that $ch_i = fd_i$ for all $i \in I$. Then let $\alpha: \coprod D_i \rightarrow C$ be the unique morphism such that $\alpha\mu_i = h_i$ for all $i \in I$. It follows that $c\alpha = f\nu$. By the unique E-M diagonal property, there exists a morphism $q: \nu(\coprod D_i) \rightarrow C$ such that $cq = f\nu_M$ and $q\nu_E = \alpha$. Therefore q is the required morphism. Because c is a

monomorphism, q_i is unique.

It is well known that whenever $f: X \rightarrow Y$ is a function and $\{D_i \subseteq X \mid i \in I\}$ a family of sets, then $f(\cup D_i) = \cup f(D_i)$. This property stated categorically is important in the relationship between M -unions and strong M -unions.

Theorem 2. Let \mathfrak{C} be an E-M category. The following are equivalent:

1) \mathfrak{C} has strong M -unions.

2) \mathfrak{C} has M -unions and E-M images distribute over M -unions. That is, let $\{d_i: D_i \rightarrow X \mid i \in I\}$ be a family of morphisms in M and let (D, \mathcal{h}) be its M -union. Then, given any morphism $f: X \rightarrow Y$ it follows that $(f\mathcal{h}(D), (f\mathcal{h})_M)$ is the M -union of $\{(fd_i)_M \mid fd_i: D_i \rightarrow Y \mid i \in I\}$.

Proof. Clearly any category that has strong M -unions also has M -unions. Therefore, to show that 1) implies 2), let $\{d_i: D_i \rightarrow X \mid i \in I\}$ be a family of morphisms in M . Let (D, \mathcal{h}) be the strong M -union of this family. By the definition of strong M -union there exists a family of morphisms $\{v_i: D_i \rightarrow D \mid i \in I\}$ such that $\mathcal{h}v_i = d_i$ for all $i \in I$.

Let $f: X \rightarrow Y$ be any morphism. By the unique E-M diagonal property, there exists for each $i \in I$ a morphism $q_i: fd_i(D_i) \rightarrow f\mathcal{h}(D)$ such that $(f\mathcal{h})_M q_i = (fd_i)_M$ and $q_i (fd_i)_E = (f\mathcal{h})_E v_i$.

Therefore, to show that $(f\mathcal{h}(D), (f\mathcal{h})_M)$ is the

M -union of $(fd_i)_M \quad i \in I$, let $c: C \rightarrow Y$ be a morphism in M and let $\{h_i: fd_i(D_i) \rightarrow C \mid i \in I\}$ be a family of morphisms such that $ch_i = (fd_i)_M$ for all $i \in I$. Since $f: X \rightarrow Y$ is a morphism, $c: C \rightarrow Y$ a morphism in M , and $\{h_i(f d_i)_E: D_i \rightarrow C \mid i \in I\}$ a family of morphism such that $ch_i(f d_i)_E = fd_i$ for all $i \in I$, it follows from the definition of strong M -union that there exists a morphism $m: D \rightarrow C$ such that $cm = fh$. By the unique E-M diagonal property, there exists a morphism $\nu: fh(D) \rightarrow C$ such that $c\nu = (fh)_M$ and $\nu(fh)_E = m$. Hence ν is the required morphism.

To show uniqueness, let $\nu, \nu^*: fh(D) \rightarrow C$ be morphisms such that $c\nu = c\nu^* = (fh)_M$. Therefore $c\nu(fh)_E = c\nu^*(fh)_E = fh$. But from the definition of strong M -union, $\nu(fh)_E = \nu^*(fh)_E$. By the unique E-M diagonal property, $\nu = \nu^*$.

To show that 2) implies 1), let $\{d_i: D_i \rightarrow X \mid i \in I\}$ be a family of morphisms in M and let (D, h) be its M -union. Let $f: X \rightarrow A$ be a morphism, $c: C \rightarrow A$ a morphism in M , and $\{h_i: D_i \rightarrow C \mid i \in I\}$ a family of morphisms such that $ch_i = fd_i$ for all $i \in I$. By the unique E-M diagonal property, there exists for each $i \in I$ a morphism $g_i: fd_i(D_i) \rightarrow C$ such that $cg_i = (fd_i)_M$ and $g_i(fd_i)_E = h_i$.

Because E-M images distribute over M -unions, it follows that $(fh(D), (fh)_M)$ is the M -union of $\{(fd_i)_M \mid i \in I\}$. Therefore there exists a morphism $q: fh(D) \rightarrow C$ such that $cq = (fh)_M$. Hence $q(fh)_E: D \rightarrow C$ is a mor-

phism such that $cq(fh)_E = fh$.

To show uniqueness, let $l, l^* : D \rightarrow C$ be morphisms such that $cl = cl^* = fh$. Applying the unique E-M diagonal property twice, we get morphisms $m, m^* : fh(D) \rightarrow C$ such that $cm = (fh)_M$, $m(fh)_E = l$, $cm^* = (fh)_M$, and $m^*(fh)_E = l^*$. From the definition of M -union it follows that $m = m^*$. Therefore $l = m(fh)_E = m^*(fh)_E = l^*$.

4. Coreflective subcategories. The only subcategories considered in this paper will be assumed to be both full and replete. That is, given K a subcategory of ξ :

1) Whenever A and B are objects in K and $f : A \rightarrow B$ is a morphism in ξ , then f must also be a morphism in K (K is full) .

2) Whenever A is an object in K and B is isomorphic to A , then B must also be an object in K (K is replete) .

Definition 3. Let K be a subcategory of ξ .

K is a coreflective subcategory of ξ if and only if for every object A in ξ , there exists an object A_K in K and a morphism $k : A_K \rightarrow A$ such that whenever B is an object in K and $f : B \rightarrow A$ is a morphism, it follows that there exists a unique morphism $q : B \rightarrow A_K$ such that $kq = f$. In this case k is the coreflection morphism of A in K .

Given a class of morphisms M , let K be a coreflec-

tive subcategory of ξ . K is an M -coreflective subcategory of ξ if and only if each coreflection morphism is a morphism in M .

Henceforth, it is assumed that M is a class of monomorphism which is closed under composition.

Proposition 2. M -coreflective subcategories are closed under M -unions. That is, if K is an M -coreflective subcategory of ξ , $\{d_i : D_i \rightarrow X \mid i \in I\}$ a family of morphisms in M where each D_i is an object in K , and (D, \mathcal{K}) the M -union of this family, then D is also an object in K .

Proof. From the definition of M -union, there exists a family of morphisms $\{v_i : D_i \rightarrow D \mid i \in I\}$ such that $\mathcal{K}v_i = d_i$ for all $i \in I$.

Let $\mathcal{K} : D_{\mathcal{K}} \rightarrow D$ be the coreflection morphism of D in K . There exists for each $i \in I$, a morphism $q_i : D_i \rightarrow D_{\mathcal{K}}$ such that $\mathcal{K}q_i = v_i$.

Hence $\mathcal{K}\mathcal{K} : D_{\mathcal{K}} \rightarrow X$ is a morphism in M and $\{q_i : D_i \rightarrow D_{\mathcal{K}} \mid i \in I\}$ a family of morphisms such that $\mathcal{K}\mathcal{K}q_i = d_i$ for all $i \in I$. By the definition of M -union, it follows that \mathcal{K} is an isomorphism.

Since K is replete, D is an object in K .

E-M factorizations are too powerful in this setting, so a simpler factorization is defined.

Definition 4. Let $f : A \rightarrow B$ be a morphism. The M -image of f is a morphism $I_f : C \rightarrow B$ in M such that:

1) There exists a morphism $e : A \rightarrow C$ such that $I_f e = f$.

2) Whenever $m: D \rightarrow B$ is a morphism in M and $k: A \rightarrow D$ a morphism such that $mk = f$, it follows that there exists a unique morphism $q: C \rightarrow D$ such that $mq = I_f$.

Remark. All categories which have coproducts and M -images have M -unions.

Proposition 5. M -coreflective subcategories are closed under M -images. That is, if K is an M -coreflective subcategory of ξ , $f: A \rightarrow B$ a morphism such that A is an object in K , and $I_f: C \rightarrow B$ the M -image of f , then C is also an object in K .

Proof. From the definition of M -image, there exists a morphism $e: A \rightarrow C$ such that $I_f e = f$. Let $k: C_K \rightarrow C$ be the coreflection morphism of C in K . Because A is an object in K , there exists a morphism $q: A \rightarrow C_K$ such that $kq = e$.

Hence, $I_f k: C_K \rightarrow B$ is a morphism in M and $q: A \rightarrow C_K$ a morphism such that $I_f kq = I_f e = f$. Therefore, from the definition of M -image, k is an isomorphism. Because K is replete, C is an object in K .

The following proposition is similar to one stated in a paper by Herrlich and Strecker [2] except that it uses M -unions and M -images rather than coproducts and extremal epimorphisms.

Theorem 3. Let ξ be an M -locally small category

that has M -unions and M -images. Let K be a subcategory of ξ . The following are equivalent:

- 1) K is an M -coreflective subcategory of ξ .
- 2) K is closed under M -unions and M -images.

Proof. That 1) implies 2) has already been shown. Therefore, to show 2) implies 1), let A be an object in ξ . Let $\{d_i : D_i \rightarrow A \mid i \in I\}$ be a representative family of M -morphisms with codomain A such that each D_i is an object in K .

Let (D, \mathcal{A}) be the M -union of the d_i . Because K is closed under M -unions, D is an object in K . It will be shown that \mathcal{A} is the coreflection morphism of A in K .

Let B be an object in K and let $f : B \rightarrow A$ be a morphism. Let $I_f : C \rightarrow A$ be the M -image of f . Because K is closed under M -images, C is an object in K . Since $I_f : C \rightarrow A$ is a morphism in M , there exists some $j \in I$ and an isomorphism $q : C \rightarrow D_j$ such that $d_j q = I_f$.

Therefore, since there exists a morphism $e : B \rightarrow C$ such that $I_f e = f$ and a family of morphisms $\{v_i : D_i \rightarrow D \mid i \in I\}$ such that $\mathcal{A} v_i = d_i$ for all $i \in I$, then $v_j q e : B \rightarrow D$ is a morphism such that $\mathcal{A} v_j q e = f$.

Because \mathcal{A} is a monomorphism, this induced morphism is unique. Thus \mathcal{A} is the coreflection morphism of A in K .

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