John A. Tiller

Unions in $E-M$ categories and coreflective subcategories

Commentationes Mathematicae Universitatis Carolinae, Vol. 15 (1974), No. 1, 173--187

Commentationes Mathematicae Universitatis Carolinae

15,1 (1974)

UNIONS IN E-M CATEGORIES AND COREFLECTIVE SUBCATEGORIES [x)]

John TILLER, Conway

Abstract: The concept of $\mathcal{M}$ -unions in categories is defined and discussed and a characterization of coreflective subcategories by means of this concept is given.

Key-words: $\mathcal{M}$ -union, $\mathcal{M}$ -image, factorization, coreflective subcategory.

AMS: 18A30, 18A40 Ref. Ž. 2.726.23

---

1. Introduction. This paper will be concerned with categorial unions in two settings. First, in an E-M category, $\mathcal{M}$ -unions will be defined and discusses. It will be shown that the definition of $\mathcal{M}$ -unions can be made stronger than the expected definition and that $\mathcal{M}$ -unions exist in many E-M categories.

Second, looking at coreflective subcategories, a characterization of $\mathcal{M}$ -coreflective subcategories will be obtained with the use of $\mathcal{M}$ -unions and $\mathcal{M}$ -images.

Categorical unions have never attracted much attention because coproducts are generally a stronger and more basic idea. However, categorical unions are the generaliza-

----

x) This research was conducted under the direction of Professor Temple H. Fay and partially supported by Arkansas Educational Research Development Council grant 036.

tion of a very intuitive concept that appears in many situations. For example, unions in the category of all topological spaces take on a simpler form than do coproducts and are more useful in applications.

The categorical definitions not stated in this paper can be found in Mitchell [4], MacLane [3], or Herrlich and Strecker [1].

2. E-M <u>category</u>. E-M categories arise naturally in all categories where some notion of images is introduced. This is stated categorically in terms of factorizations of morphisms.

<u>Definition 1.</u> Let $\mathcal{C}$ be a category and let E and $\mathcal{M}$ be classes of morphisms which are closed under composition with all isomorphisms. We call $\mathcal{C}$ an E-M <u>category</u> if and only if:

1) Every morphism in $\mathcal{C}$ has an E-M <u>factorization</u>. That is, given a morphism $f : A \to B$, there exist morphisms $e : A \to C$ and $m : A \to C$ with $e \in E$ and $m \in \mathcal{M}$ such that $me = f$.

2) $\mathcal{C}$ has the <u>unique</u> E-M <u>diagonal property</u>. That is, given a commutative square $mg = fe$ with $e \in E$ and $m \in \mathcal{M}$, there exists a unique morphism q such that $mq = f$ and $qe = g$.

<u>Examples</u>. Any category is an E-M category, where E is the class of all morphisms (all isomorphisms) and $\mathcal{M}$ is the class of all isomorphisms (resp. all morphisms).

The categories of all sets, semigroups, monoids, groups, Abelian groups, rings, commutative rings, and compact Hausdorff spaces are E-M categories where E is the class of all surjective morphisms and M is the class of all injective morphisms.

The categories of all topological spaces, Hausdorff spaces, compact spaces, and connected spaces are E-M categories, where E is the class of all dense maps (surjective maps, quotient maps) and M is the class of all closed embeddings (resp. embeddings, injective maps).

The categories of all topological spaces and all Hausdorff spaces are E-M categories, where E is the class of all final maps and M is the class of all bijective maps.

The categories of all topological spaces, compact spaces, and connected spaces are E-M categories, where E is the class of all bijective maps and M is the class of all cofinal maps.

It follows from the definition that, in an E-M category, E-M factorizations are essentially unique. Therefore, given a morphism $g: A \to B$ in an E-M category, $g_E : A \to g(A)$ and $g_M : g(A) \to B$ will denote the essentially unique E-M factorization of g .

3. M -union. M -unions are a generalization of usual categorical unions.

Definition 2. Let M be a class of morphisms and let $\{d_i : D_i \to X \mid i \in I\}$ be a family of morphisms in M . Let (D, h) be a pair, where D is an object and $h : D \to X$ is a morphism in $\mathcal{M}$ such that there exists a family of morphisms $\{v_i : D_i \longrightarrow D \mid i \in I\}$ for which $hv_i = d_i$ for all $i \in I$.

We say (D, h) is the $\mathcal{M}$ -union of $\{d_i \mid i \in I\}$ if and only if

(U) Whenever $c : C \longrightarrow X$ is a morphism in $\mathcal{M}$ and $\{k_i : D_i \longrightarrow C \mid i \in I\}$ a family of morphisms such that $ck_i = d_i$ for all $i \in I$, it follows that there exists a unique morphism $q : D \longrightarrow C$ such that $cq = h$.

$$\begin{tikzcd}
C \arrow[rrd, "c"] & & \\
& D \arrow[ul, dashed, "q"'] \arrow[r, "h"] & X \\
D_i \arrow[uu, "k_i"] \arrow[ur, "v_i"] \arrow[urr, "d_i"'] & &
\end{tikzcd}$$

We say (D, h) is the strong $\mathcal{M}$ -union of $\{d_i \mid i \in I\}$ if and only if

(SU) Whenever $f : X \longrightarrow A$ is a morphism, $c : C \longrightarrow A$ a morphism in $\mathcal{M}$, and $\{k_i : D_i \longrightarrow C \mid i \in I\}$ a family of morphisms such that $fd_i = ck_i$ for all $i \in I$, it follows that there exists a unique morphism $q : D \longrightarrow C$ such that $cq = fh$.

$$
\begin{tikzcd}
C \arrow[rr, "c"] & & A \\
& D \arrow[ul, dashed, "q"'] \arrow[r, "h"] & X \arrow[u, "f"'] \\
D_i \arrow[uu, "k_i"] \arrow[ur, "v_i"] \arrow[urr, "d_i"'] & &
\end{tikzcd}
$$

Strong $\mathcal{M}$-unions are more useful in E-M categories while $\mathcal{M}$-unions suffice in other settings (such as coreflective subcategories).

Although the two unions differ by definition, they coincide in E-M categories under very weak hypothesis. More precisely:

Theorem 1. In an E-M category that has weak pullbacks, let $\{d_i : D_i \longrightarrow X \mid i \in I\}$ be a family of morphism in $\mathcal{M}$. Let $h : D \longrightarrow X$ be a morphism in $\mathcal{M}$ through whic each d_i factors. Then the following are equivalent:

1) (D, h) is the strong $\mathcal{M}$-union of $\{d_i \mid i \in I\}$.

2) (D, h) is the $\mathcal{M}$-union of $\{d_i \mid i \in I\}$.

Proof. That 1) implies 2) is clear by setting $f = 1_X$ in the definition of strong $\mathcal{M}$-union.

To show 2) implies 1), let $f : X \longrightarrow A$ be a morphism $c : C \longrightarrow A$ a morphism in $\mathcal{M}$, and $\{k_i : D_i \longrightarrow C \mid i \in I\}$ a family of morphisms such that $ck_i = fd_i$ for all $i \in I$. Then let the following diagram be a weak pullback diagram.

$$\begin{tikzcd} C \arrow[r, "c"] & A \\ P \arrow[u, "p"] \arrow[r, "b"'] & X \arrow[u, "f"'] \end{tikzcd}$$

By the unique E-M diagonal property, there exists a morphism $q : b(P) \longrightarrow C$ such that $cq = f b_M$ and $q b_E = p$. Therefore the following is a weak pullback diagram.

$$\begin{tikzcd} C \arrow[r, "c"] & A \\ b(P) \arrow[u, "q"] \arrow[r, "b_M"'] & X \arrow[u, "f"'] \end{tikzcd}$$

Since $ck_i = fd_i$, from the definition of weak pullback there exists for each $i \in I$ a morphism $z_i : D_i \longrightarrow b(P)$ such that $qz_i = k_i$ and $b_M z_i = d_i$

Hence, from the hypothesis, there exists a morphism $x : D \longrightarrow b(P)$ such that $b_M x = h$. Therefore $qx : D \longrightarrow C$ is a morphism such that $cqx = f b_M x = fh$.

To show uniqueness, let $m, m^* : D \longrightarrow C$ be morphisms such that $cm = cm^* = fh$. From the definition of weak pullback, there exist morphisms $d, d^* : D \longrightarrow b(P)$ such that $qd = m$, $b_M d = h$, $qd^* = m^*$, and $b_M d^* = h$. But from the hypothesis, $d = d^*$. Therefore $m = = qd = qd^* = m^*$.

<u>Examples</u>. In the category of all sets, let M be the class of all injective functions. Given a family of sets $\{D_i \subseteq X \mid i \in I\}$, the M-union of their inclusions $d_i : D_i \longrightarrow X$ is the pair $(\cup D_i, h)$, where $\cup D_i$ is the usual

set-theoretic union and $h : \cup D_i \longrightarrow X$ is the inclusion function.

In the category of all groups, let $\mathcal{M}$ be the class of all injective homomorphisms. Given a family of subgroups $\{D_i \mid i \in I\}$ of the group X, the $\mathcal{M}$-union of their inclusion functions $d_i : D_i \longrightarrow X$ is the pair $(\langle\{D_i\}\rangle, h)$, where $\langle\{D_i\}\rangle$ is the subgroup generated by the subgroups D_i and $h : \langle\{D_i\}\rangle \longrightarrow X$ is the inclusion homomorphism.

In the category of all topological spaces, let X be a topological space and consider a family of spaces $\{D_i \mid i \in I\}$, where each set D_i is a subset of the set X.

1) When $\mathcal{M}$ is the class of all embeddings and each inclusion $d_i : D_i \longrightarrow X$ is an embedding, the $\mathcal{M}$-union of the d_i is the pair $(\cup D_i, h)$, where $\cup D_i$ is the set-theoretic union of the sets D_i. Here $\cup D_i$ is endowed with the subspace topology and $h : \cup D_i \longrightarrow X$ is the inclusion map.

2) When $\mathcal{M}$ is the class of all injective maps and each inclusion $d_i : D_i \longrightarrow X$ is an injective map, the $\mathcal{M}$-union of the d_i is the pair $(\cup D_i, h)$, where $\cup D_i$ is the set-theoretic union of the D_i. Here $\cup D_i$ is endowed with the topology defined by the following:

A subset O is open in $\cup D_i$ if and only if $O \cap D_i$ is open in D_i for all $i \in I$.

The map $h : \cup D_i \longrightarrow X$ is the inclusion map.

3) Let $\mathcal{M}$ be the class of all closed embeddings and each inclusion $d_i : D_i \longrightarrow X$ a closed embedding. Then

the $\mathcal{M}$-union of the d_i is the pair $(cl(\cup D_i), h)$ where $cl(\cup D_i)$ is the closure of the set-theoretic union of the D_i. Here $cl(\cup D_i)$ is endowed with the subspace topology and $h : cl(\cup D_i) \longrightarrow X$ is the inclusion map.

For an arbitrary E-M category $\mathcal{C}$, it is next shown that the existence of strong $\mathcal{M}$-unions is guaranteed when $\mathcal{C}$ has coproducts and $\mathcal{M}$ consists entirely of monomorphisms.

Proposition 1. In an E-M category where $\mathcal{M}$ is a class of monomorphisms, let $\{d_i : D_i \longrightarrow X \mid i \in I\}$ be a family of morphisms in $\mathcal{M}$. Let the family of morphisms $\{u_i : D_i \longrightarrow \amalg D_i \mid i \in I\}$ be the coproduct of the D_i. Furthermore, let $p : \amalg D_i \longrightarrow X$ be the unique morphism guaranteed by the definition of coproduct such that $pu_i = d_i$ for all $i \in I$. It then follows that $(p(\amalg D_i), p_M)$ is the strong $\mathcal{M}$-union of the d_i.

Proof. First, there exists the family of morphisms $\{p_E u_i : D_i \longrightarrow p(\amalg D_i) \mid i \in I\}$ such that $p_M p_E u_i = pu_i = d_i$ for all $i \in I$.

Second, let $f : X \longrightarrow A$ be a morphism, $c : C \longrightarrow A$ a morphism in $\mathcal{M}$, and $\{k_i : D_i \longrightarrow C \mid i \in I\}$ a family of morphisms such that $ck_i = fd_i$ for all $i \in I$. Then let $x : \amalg D_i \longrightarrow C$ be the unique morphism such that $xu_i = k_i$ for all $i \in I$. It follows that $cx = fp$. By the unique E-M diagonal property, there exists a morphism $q : p(\amalg D_i) \longrightarrow C$ such that $cq = fp_N$ and $qp_E = x$. Therefore q is the required morphism. Because c is a monomorphism, q is unique.

It is well known that whenever $f : X \longrightarrow Y$ is a function and $\{D_i \subseteq X \mid i \in I\}$ a family of sets, then $f(\cup D_i) = \cup f(D_i)$. This property stated categorically is important in the relationship between M-unions and strong M-unions.

<u>Theorem 2</u>. Let $\mathfrak{k}$ be an E-M category. The following are equivalent:

1) $\mathfrak{k}$ has strong M-unions.

2) $\mathfrak{k}$ has M-unions and E-M <u>images distribute over M-unions</u>. That is, let $\{d_i : D_i \longrightarrow X \mid i \in I\}$ be a family of morphisms in M and let (D, h) be its M-union. Then, given any morphism $f : X \longrightarrow Y$ it follows that $(fh(D), (fh)_M)$ is the M-union of $\{(fd_i)_M :$ $: fd_i(D_i) \longrightarrow Y \mid i \in I\}$.

<u>Proof</u>. Clearly any category that has strong M-unions also has M-unions. Therefore, to show that 1) implies 2), let $\{d_i : D_i \longrightarrow X \mid i \in I\}$ be a family of morphisms in M. Let (D, h) be the strong M-union of this family. By the definition of strong M-union there exists a family of morphisms $\{v_i : D_i \longrightarrow D \mid i \in I\}$ such that $hv_i = d_i$ for all $i \in I$.

Let $f : X \longrightarrow Y$ be any morphism. By the unique E-M diagonal property, there exists for each $i \in I$ a morphism $q_i : fd_i(D_i) \longrightarrow fh(D)$ such that $(fh)_M q_i = (fd_i)_M$ and $q_i(fd_i)_E = (fh)_E v_i$.

Therefore, to show that $(fh(D), (fh)_M)$ is the

M -union of $(fd_i)_M$ $i \in I$, let $c: C \to Y$ be a morphism in M and let $\{k_i : fd_i(D_i) \to C \mid i \in I\}$ be a family of morphisms such that $ck_i = (fd_i)_M$ for all $i \in I$. Since $f: X \to Y$ is a morphism, $c: C \to Y$ a morphism in M, and $\{k_i (fd_i)_E : D_i \to C \mid i \in I\}$ a family of morphism such that $ck_i(fd_i)_E = fd_i$ for all $i \in I$, it follows from the definition of strong M -union that there exists a morphism $m: D \to C$ such that $cm = fh$. By the unique E-M diagonal property, there exists a morphism $p: fh(D) \to C$ such that $cp = (fh)_M$ and $p(fh)_E = m$. Hence p is the required morphism.

To show uniqueness, let $b, b^* : fh(D) \to C$ be morphisms such that $cb = cb^* = (fh)_M$. Therefore $cb(fh)_E = cb^*(fh)_E = fh$. But from the definition of strong M -union, $b(fh)_E = b^*(fh)_E$. By the unique E-M diagonal property, $b = b^*$.

To show that 2) implies 1), let $\{d_i : D_i \to X \mid i \in I\}$ be a family of morphisms in M and let (D, h) be its M -union. Let $f: X \to A$ be a morphism, $c: C \to A$ a morphism in M, and $\{k_i : D_i \to C \mid i \in I\}$ a family of morphisms such that $ck_i = fd_i$ for all $i \in I$. By the unique E-M diagonal property, there exists for each $i \in I$ a morphism $q_i : fd_i(D_i) \to C$ such that $cq_i = (fd_i)_M$ and $q_i(fd_i)_E = k_i$.

Because E-M images distribute over M -unions, it follows that $(fh(D), (fh)_M)$ is the M -union of $\{(fd_i)_M \mid i \in I\}$. Therefore there exists a morphism $q: fh(D) \to C$ such that $cq = (fh)_M$. Hence $q(fh)_E : D \to C$ is a morphism such that $cq(fh)_E = fh$.

To show uniqueness, let $b, b^*: D \longrightarrow C$ be morphisms such that $cb = cb^* = fh$. Applying the unique E-M diagonal property twice, we get morphisms $m, m^*: fh(D) \longrightarrow C$ such that $cm = (fh)_M$, $m(fh)_E = b$, $cm^* = (fh)_M$, and $m^*(fh)_E = b^*$. From the definition of $\mathcal{M}$-union it follows that $m = m^*$. Therefore $b = m(fh)_E =$
$= m^*(fh)_E = b^*$.

4. Coreflective subcategories. The only subcategories considered in this paper will be assumed to be both full and replete. That is, given $\mathcal{K}$ a subcategory of $\mathcal{G}$:

1) Whenever A and B are objects in $\mathcal{K}$ and $f: A \longrightarrow B$ is a morphism in $\mathcal{G}$,then f must also be a morphism in $\mathcal{K}$ ($\mathcal{K}$ is full) .

2) Whenever A is an object in $\mathcal{K}$ and B is isomorphic to A , then B must also be an object in $\mathcal{K}$ ($\mathcal{K}$ is replete).

Definition 3. Let $\mathcal{K}$ be a subcategory of $\mathcal{G}$.

$\mathcal{K}$ is a coreflective subcategory of $\mathcal{G}$ if and only if for every object A in $\mathcal{G}$, there exists an object A_K in $\mathcal{K}$ and a morphism $k: A_K \longrightarrow A$ such that whenever B is an object in $\mathcal{K}$ and $f: B \longrightarrow A$ is a morphism, it follows that there exists a unique morphism $q: B \longrightarrow A_K$ such that $kq = f$. In this case k is the coreflection morphism of A in $\mathcal{K}$.

Given a class of morphisms $\mathcal{M}$, let $\mathcal{K}$ be a coreflective subcategory of $\mathcal{G}$. $\mathcal{K}$ is an $\mathcal{M}$ -coreflective subcategory of $\mathcal{G}$ if and only if each coreflection morphism is a morphism in $\mathcal{M}$.

Henceforth, it is assumed that $\mathcal{M}$ is a class of monomorphism which is closed under composition.

Proposition 2. $\mathcal{M}$ -coreflective subcategories are closed under $\mathcal{M}$ -unions. That is, if $\mathcal{K}$ is an $\mathcal{M}$ -coreflective subcategory of $\mathcal{G}$, $\{d_i : D_i \longrightarrow X \mid i \in I\}$ a family of morphisms in $\mathcal{M}$ where each D_i is an object in $\mathcal{K}$, and (D, h) the $\mathcal{M}$ -union of this family, then D is also an object in $\mathcal{K}$.

Proof. From the definition of $\mathcal{M}$ -union, there exists a family of morphisms $\{v_i : D_i \longrightarrow D \mid i \in I\}$ such that $hv_i = d_i$ for all $i \in I$.

Let $k : D_k \longrightarrow D$ be the coreflection morphism of D in $\mathcal{K}$. There exists for each $i \in I$, a morphism $q_i : D_i \longrightarrow D_k$ such that $kq_i = v_i$.

Hence $hk : D_K \longrightarrow X$ is a morphism in $\mathcal{M}$ and $\{q_i : D_i \longrightarrow D_K \mid i \in I\}$ a family of morphisms such that $hkq_i = d_i$ for all $i \in I$. By the definition of $\mathcal{M}$ -union, it follows that k is an isomorphism.

Since $\mathcal{K}$ is replete, D is an object in $\mathcal{K}$.

E-M factorizations are too powerful in this setting, so a simpler factorization is defined.

Definition 4. Let $f : A \longrightarrow B$ be a morphism. The $\mathcal{M}$ -image of f is a morphism $I_f : C \longrightarrow B$ in $\mathcal{M}$ such that:

1) There exists a morphism $e : A \longrightarrow C$ such that $I_f e = f$.

2) Whenever $m : D \longrightarrow B$ is a morphism in $\mathcal{M}$ and $h :$ $: A \longrightarrow D$ a morphism such that $mh = f$, it follows that there exists a unique morphism $q : C \longrightarrow D$ such that $mq = I_f$.

Remark. All categories which have coproducts and $\mathcal{M}$ -images have $\mathcal{M}$ -unions.

Proposition 5. $\mathcal{M}$ -coreflective subcategories are closed under $\mathcal{M}$ -images. That is, if $\mathcal{K}$ is an $\mathcal{M}$-coreflective subcategory of $\mathcal{C}$, $f : A \longrightarrow B$ a morphism such that A is an object in $\mathcal{K}$, and $I_f : C \longrightarrow B$ the $\mathcal{M}$-image of f , then C is also an object in $\mathcal{K}$.

Proof. From the definition of $\mathcal{M}$ -image, there exists a morphism $e : A \longrightarrow C$ such that $I_f e = f$. Let $h :$ $: C_K \longrightarrow C$ be the coreflection morphism of C in $\mathcal{K}$. Because A is an object in $\mathcal{K}$, there exists a morphism $q : A \longrightarrow C_K$ such that $hq = e$.

Hence, $I_f h : C_K \longrightarrow B$ is a morphism in $\mathcal{M}$ and $q : A \longrightarrow$ $\longrightarrow C_K$ a morphism such that $I_f hq = I_f e = f$. Therefore, from the definition of $\mathcal{M}$ -image, h is an isomorphism. Because $\mathcal{K}$ is replete, C is an object in $\mathcal{K}$.

The following proposition is similar to one stated in a paper by Herrlich and Strecker [2] except that it uses $\mathcal{M}$ -unions and $\mathcal{M}$ -images rather than coproducts and extremal epimorphisms.

Theorem 3. Let $\mathcal{C}$ be an $\mathcal{M}$ -locally small category that has $\mathcal{M}$ -unions and $\mathcal{M}$ -images. Let $\mathcal{K}$ be a subcategory of $\mathcal{G}$. The following are equivalent:

1) $\mathcal{K}$ is an $\mathcal{M}$ -coreflective subcategory of $\mathcal{G}$.

2) $\mathcal{K}$ is closed under $\mathcal{M}$ -unions and $\mathcal{M}$ -images.

Proof. That 1) implies 2) has already been shown. Therefore, to show 2) implies 1), let A be an object in $\mathcal{G}$. Let $\{ d_i : D_i \longrightarrow A \mid i \in I \}$ be a representative family of $\mathcal{M}$ -morphisms with codomain A such that each D_i is an object in $\mathcal{K}$.

Let (D, h) be the $\mathcal{M}$ -union of the d_i . Because $\mathcal{K}$ is closed under $\mathcal{M}$ -unions, D is an object in $\mathcal{K}$. It will be shown that h is the coreflection morphism of A in $\mathcal{K}$.

Let B be an object in $\mathcal{K}$ and let $f : B \longrightarrow A$ be a morphism. Let $I_f : C \longrightarrow A$ be the $\mathcal{M}$ -image of f . Because $\mathcal{K}$ is closed under $\mathcal{M}$ -images, C is an object in $\mathcal{K}$. Since $I_f :$ $: C \longrightarrow A$ is a morphism in $\mathcal{M}$, there exists some $j \in I$ and an isomorphism $q : C \longrightarrow D_j$ such that $d_j q = I_f$.

Therefore, since there exists a morphism $e : B \longrightarrow C$ such that $I_f e = f$ and a family of morphisms $\{ v_i : D_i \longrightarrow D \mid$ $\mid i \in I \}$ such that $h v_i = d_i$ for all $i \in I$, then $v_j q e :$ $: B \longrightarrow D$ is a morphism such that $h v_j q e = f$.

Because h is a monomorphism, this induced morphism is unique. Thus h is the coreflection morphism of A in $\mathcal{K}$.

R e f e r e n c e s

[1] H. HERRLICH and G.E. STRECKER: Category Theory, Allyn and Bacon, Boston, 1973.

[2] H. HERRLICH and G.E. STRECKER: Coreflective subcategories, Trans.Amer.Math.Soc.157(1971), 205-226.

[3] S. MacLANE: Categories for the Working Mathematician, Springer-Verlag,New York,1971.

[4] B. MITCHELL: Theory of Categories, Academic Press, New York,1965.

Hendrix College (sophomore)

Conway

Arkansas, U.S.A.

(Oblatum 28.8.1973)