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STRONG EMBEDDINGS INTO CATEGORIES OF ALGEBRAS OVER A MONAD

II.

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Abstract: Hedrlín, Isbell, Kučera, Pultr, Trnková and others have intensely investigated full and strong embeddings of concrete categories into categories of algebras. This paper considers the possibility of replacing usual categories of algebras by equational and varietal categories in the sense of Linton. All considerations are carried out for an arbitrary category in the place of the category of sets.

Key-words: Equational category, varietal category, U-algebra, monad, algebra over a monad, full embedding, strong embedding, Kan extension, Beck's theorem, absolute limit, split coequalizer.

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This is the second part of the paper which appeared in this journal in 1973.

§ 3. Reflection of limits and colimits

Lemma 2. Let (M, U) be structured over A, (N, W) over B, $F: A \longrightarrow B$ a functor and $H: M \longrightarrow N$ an F-nice embedding. Let J be a category, $D: J \longrightarrow M$ a functor, $m \in M$ and $\nu: m \longrightarrow D$ a cone to the base D from the vertex m (i.e. a natural transformation from the constant functor with the value m to D) such that

$\mathcal{U}\nu : \mathcal{U}m \xrightarrow{\cdot} \mathcal{U}D$, $H\nu : Hm \xrightarrow{\cdot} HD$ and $F\mathcal{U}\nu : F\mathcal{U}m \xrightarrow{\cdot} F\mathcal{U}D$ are limiting cones. Then ν is a limiting cone, too.

Proof: Let $x \in M$ and $\tau : x \xrightarrow{\cdot} D$ be a cone to D from x . Since $\mathcal{U}\nu$, $H\nu$ are limiting cones, there exist unique arrows $t_1 : \mathcal{U}x \longrightarrow \mathcal{U}m$ in A and $t_2 : Hx \longrightarrow Hm$ in N with $\mathcal{U}\tau_i = \mathcal{U}\nu_i \cdot t_1$ and $H\tau_i = H\nu_i \cdot t_2$ for any $i \in J$. Since $F\mathcal{U}\nu$ is a limiting cone, one gets that $Ft_1 = Wt_2$ .Now, from the fact that H is F -nice we obtain an arrow $t : x \longrightarrow m$ in M such that $Ht = t_2$. Finally, $\nu_i t = \tau_i$ because H is faithful and this equality determines t uniquely by the same argument.

Before stating the following theorem we recall that an absolute colimit is a colimit which is preserved by any functor whatever. Let $f, g : a \longrightarrow b$ be two arrows in A . An arrow $e : b \longrightarrow c$ in A is called a split coequalizer of f and g if there exist arrows $s : c \longrightarrow b$ and $t : b \longrightarrow a$ in A such that the following conditions are fulfilled: $ef = eg$, $es = id_c$, $ft = id_b$, $gt = se$. Any split coequalizer is an absolute coequalizer (see [14]). If $h : a \longrightarrow b$, $g : b \longrightarrow a$ are in A and $gh = id_a$, then g is called a split epi and h a split monic. Of course, g is epi and h monic.

Theorem 3. Let $(M, \mathcal{U})$ be structured over A , $F : A \longrightarrow B$ and there exist an F-nice embedding $H : M \longrightarrow V\text{-}Alg$ for some $V : X \longrightarrow B$. Then

a) $\mathcal{U}$ reflects limits which are preserved by F

b) $\mathcal{U}$ reflects colimits which are preserved by F

and F^{m} for each $m \in B$

c) $\mathcal{U}$ has the property

if $a, b, c \in \mathcal{M}$, $f: c \to a$, $g: c \to b$ in $\mathcal{M}$, $\mathcal{U}f$ is a split epi and

$(*)$ $\mathcal{U}g = h\mathcal{U}(f)$ for an arrow $h: \mathcal{U}a \to \mathcal{U}b$ in $\mathcal{A}$, then there is an $h': a \to b$ in $\mathcal{M}$ such that $\mathcal{U}h' = h$

d) $\mathcal{U}$ has the property

if $a, b, c \in \mathcal{M}$, $f: a \to c$, $g: b \to c$ in $\mathcal{M}$, $\mathcal{U}g$ is a split monic and

$(*)^{op}$ $\mathcal{U}f = \mathcal{U}(g)h$ for an arrow $h: \mathcal{U}a \to \mathcal{U}b$ in $\mathcal{A}$, then there is an $h': a \to b$ in $\mathcal{M}$ such that $\mathcal{U}h' = h$.

Proof: a) $|\ |_V$ creates limits for any category $V\text{-}\mathcal{A}lg$ (see [12] § 6). Let $\lambda: x \xrightarrow{\cdot} D$ be a cone to $D: \mathcal{J} \to V\text{-}\mathcal{A}lg$ from $x \in V\text{-}\mathcal{A}lg$ for which $|\ |_V \lambda: |\ |_V x \xrightarrow{\cdot} |\ |_V D$ is a limiting cone. Let $\tau:$ $: y \xrightarrow{\cdot} D$ be the created limiting cone in $V\text{-}\mathcal{A}lg$, i.e. $|\ |_V \tau = |\ |_V \lambda$. Hence, there exists a unique V-homomorphism $t: x \to y$ with $\lambda_i = \tau_i t$ for each $i \in \mathcal{J}$. Moreover, $|\ |_V t = id_{|\ |_V x}$ and therefore t is an isomorphism. Thus $|\ |_V$ reflects limits.

Let $\nu: m \xrightarrow{\cdot} D$ be a cone to $D: \mathcal{J} \to \mathcal{M}$ from $m \in \mathcal{M}$ for which $\mathcal{U}\nu: \mathcal{U}m \to \mathcal{U}D$ and $F\mathcal{U}\nu: F\mathcal{U}m \xrightarrow{\cdot}$ $\xrightarrow{\cdot} F\mathcal{U}D$ are limiting cones. $H\nu: Hm \xrightarrow{\cdot} HD$ is a limiting cone because $|\ |_V$ reflects limits and Lemma 2 asserts that $\nu: m \to D$ is a limiting cone. Hence $\mathcal{U}$

reflects limits which are preserved by F .

b) At first, we shall show that $|\ |_V : V\text{-}\mathcal{Alg} \longrightarrow B$ reflects colimits which are preserved by $(Id_B)^n$ for each $n \in B$. Let $\nu : D \dot{\longrightarrow} (a, \mathcal{A})$ be a cone from $D : J \longrightarrow V\text{-}\mathcal{Alg}$, $Di = (a_i, \mathcal{A}^i)$ for $i \in J$, to $(a, \mathcal{A}) \in V\text{-}\mathcal{Alg}$ for which $|\ |_V \nu : a_i \longrightarrow a$ and $(|\ |_V)^n \nu : a_i^n \longrightarrow a^n$ are colimiting cones. To prove ν colimiting, consider any other cone $\tau : D \dot{\longrightarrow} (b, \mathcal{B})$ from D to $(b, \mathcal{B}) \in V\text{-}\mathcal{Alg}$. Then there is a unique arrow $t : a \longrightarrow b$ in B such that $\tau_i = t \cdot \nu_i$ for each $i \in J$. Take $i \in J$, $n, k \in B$, $\theta : V^n \dot{\longrightarrow} V^k$ and consider the diagram

$$\begin{tikzcd}
a_i^n \arrow[rrrr, "\nu_i^n"] \arrow[dr, "\mathcal{A}^i_{n,k}(\theta)"] \arrow[dddrr, "\tau_i^n"'] & & & & a^n \arrow[dl, "\mathcal{A}_{n,k}(\theta)"'] \arrow[dddll, "t^n"] \\
& a_i^k \arrow[rr, "\nu_i^k"] \arrow[dr, "\tau_i^n"'] & & a^k \arrow[dl, "t^k"] & \\
& & b & & \\
& & b^n \arrow[u, "\mathcal{B}_{n,k}(\theta)"'] & &
\end{tikzcd}$$

Both triangles commute, the top and the left hand trapezoids commute by the definition of a V-homomorphism, too. Hence $t^k \cdot \mathcal{A}_{n,k}(\theta)\nu_i^n = \mathcal{B}_{n,k}(\theta) \cdot t^n \cdot \nu_i^n$ for any $i \in J$. Since ν_i^n form a colimiting cone, it holds $t^k \cdot \mathcal{A}_{n,k}(\theta) = \mathcal{B}_{n,k}(\theta)\, t^n$ and thus $t :$ $: (a, \mathcal{A}) \longrightarrow (b, \mathcal{B})$ is a V-homomorphism. Hence ν is a colimiting cone.

Now, b) follows from the dual of Lemma 2.

c) Any category $\mathcal{V}\text{-}\mathcal{Alg}$ has the property ($*$). Namely, let $(a, \mathcal{a})$, $(b, \mathcal{b})$ and $(c, \mathcal{c})$ be $\mathcal{V}$-algebras, $f : c \longrightarrow a$ and $g : c \longrightarrow b$ $\mathcal{V}$-homomorphisms, f a split epi in $\mathbf{B}$ and $g = hf$ for an arrow $h : a \longrightarrow b$ in $\mathbf{B}$. Let $n, k \in \mathbf{B}$ and $\theta : V^{n} \stackrel{\cdot}{\longrightarrow} V^{k}$. It holds $h^{k} \cdot \mathcal{a}_{n,k}(\theta) \cdot f^{n} = h^{k} \cdot f^{k} \cdot \mathcal{c}_{n,k}(\theta) = g^{k} \cdot \mathcal{c}_{n,k}(\theta) = \mathcal{b}_{n,k}(\theta) \cdot$ $\cdot g^{n} = \mathcal{b}_{n,k}(\theta) \cdot h^{n} \cdot f^{n}$. Since f is a split epi, f^{n} is a split epi and thus $h^{k} \cdot \mathcal{a}_{n,k}(\theta) = \mathcal{b}_{n,k}(\theta) \cdot h^{n}$. Hence h is a $\mathcal{V}$-homomorphism.

Let a, b, c, f, g and h be from ($*$). We have to find h' with $\mathcal{U}h' = h$. Since $F\mathcal{U}f$ is a split epi, there exists a $\mathcal{V}$-homomorphism $h_1 : Ha \longrightarrow Hb$ such that $Fh = |\,|_{\mathcal{V}} h_1$. Since H is an F-nice embedding, there exists $h' : a \longrightarrow b$ in $\mathbf{M}$ with $Hh' = h_1$. It holds $|\,|_{\mathcal{V}} H(h'f) = |\,|_{\mathcal{V}}(h_1 H(f)) = F(h) F\mathcal{U}(f) = F(h\,\mathcal{U}(f)) = F\mathcal{U}(g) =$ $= |\,|_{\mathcal{V}} H(g)$. Since $|\,|_{\mathcal{V}} H$ is faithful, it holds $h'f =$ $= g$, i.e. $\mathcal{U}(h')\,\mathcal{U}(f) = \mathcal{U}g = h\,\mathcal{U}(f)$. Hence $\mathcal{U}h' =$ $= h$ because $\mathcal{U}f$ is epi.

d) Analogously.

In particular, $\mathcal{U}$ reflects absolute limits and colimits whenever $(\mathbf{M}, \mathcal{U})$ is nicely embeddable into some $\mathcal{V}\text{-}\mathcal{Alg}$. If $\mathcal{V}\text{-}\mathcal{Alg} = \mathbf{B}^{\mathbf{T}}$ for a monad $\mathbf{T}$ in $\mathbf{B}$, then $\mathcal{U}$ reflects colimits which are preserved by F and $\mathbf{T}F$ and in d) it suffices to suppose that $\mathcal{U}g$ is monic. In the case $\mathbf{B} = \mathbf{Ens}$ in b) we can confine ourselves to the sets n for which $\mathrm{card}\ n \leq \mathrm{rank}\ \Phi_{F\mathcal{U}} \mathbf{M}$. If $\mathbf{A} = \mathbf{B} = \mathbf{Ens}$, then c) and d) are proved in [19] 3.14 for strong embeddings.

Theorem 4. Let $(\mathcal{M}, \mathcal{U})$ be structured over $\mathcal{A}$ and $\mathcal{U}$ have a left adjoint P. Then the following conditions are equivalent:

(i) $\overline{\mathcal{U}} : \mathcal{M} \longrightarrow \mathcal{A}^{\mathcal{U}P}$ is a realization

(ii) $\mathcal{M}$ is strongly embeddable into $V\text{-}\mathcal{A}lg$ for some $V : \mathcal{X} \longrightarrow \mathcal{B}$

(iii) $\mathcal{M}$ is nicely embeddable into $V\text{-}\mathcal{A}lg$ for some $V : \mathcal{X} \longrightarrow \mathcal{B}$

(iv) $\mathcal{U}$ has the property $(*)$ from Theorem 3

(v) $\mathcal{U}$ reflects split coequalizers (i.e. $\cdot \underset{\mathcal{U}g}{\overset{\mathcal{U}f}{\rightrightarrows}} \cdot \xrightarrow{\mathcal{U}h} \cdot$ a split coequalizer implies $\cdot \underset{g}{\overset{f}{\rightrightarrows}} \cdot \xrightarrow{h} \cdot$ a coequalizer).

Proof: Clearly (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and by Theorem 3 (iii) $\Longrightarrow$ (iv). Since $\cdot \underset{\mathcal{U}g}{\overset{\mathcal{U}f}{\rightrightarrows}} \cdot \xrightarrow{\mathcal{U}h} \cdot$ a split coequalizer implies that $\mathcal{U}h$ is a split epi, the implication (iv) $\Longrightarrow$ (v) is true. It remains to show that (v) $\Longrightarrow$ (i). This assertion is a part of the Beck's tripleability theorem which proof can be found in [14]. Following considerations in [14], we shall draw the proof of our implication; full details are given in 14 . At first, $\mathcal{U}$ has a codensity monad $\mathbb{R} = \mathbb{R}_{\mathcal{U}} = \mathcal{U}P$ (the monad defined by the adjunction) and $\overline{\mathcal{U}}$ is the usual comparison functor for which $\mathcal{U} = G^{\mathbb{R}}\overline{\mathcal{U}}$ and $F^{\mathbb{R}} = \overline{\mathcal{U}}P$ holds. Let $\eta : Id_{\mathcal{A}} \xrightarrow{\cdot} \mathcal{U}P$ and $\varepsilon : P\mathcal{U} \xrightarrow{\cdot} Id_{\mathcal{M}}$ be the unit and counit of the adjunction $\mathcal{M} \underset{P}{\overset{\mathcal{U}}{\rightleftarrows}} \mathcal{A}$. Let $m \in \mathcal{M}$ and consider the diagram

$$(1) \qquad PUPUm \underset{PU\varepsilon_m}{\overset{\varepsilon_{PUm}}{\rightrightarrows}} PUm \xrightarrow{\varepsilon_m} m$$

It holds $\varepsilon_m \cdot \varepsilon_{PUm} = \varepsilon_m \cdot PU\varepsilon_m$ by the naturality of ε. If the functor U is applied to this diagram, we obtain

$$UPUPUm \underset{UPU\varepsilon_m}{\overset{U\varepsilon_{PUm}}{\rightrightarrows}} UPUm \xrightarrow{U\varepsilon_m} Um$$

which is a coequalizer in A split by

$$UPUPUm \xleftarrow[\eta_{UPUm}]{} UPUm \xleftarrow[\eta_{Um}]{} Um \ .$$

By (v) (1) is a coequalizer in M.

Now, consider $f: Um \longrightarrow Um'$ such that $f: \overline{U}m \longrightarrow \overline{U}m'$ is an R-homomorphism, where $m, m' \in M$. In the diagram

$$(2) \qquad \begin{tikzcd}
PUPUm \arrow[r, shift left, "\varepsilon_{PUm}"] \arrow[r, shift right, "PU\varepsilon_m"'] \arrow[d, "PUPf"] & PUm \arrow[r, "\varepsilon_m"] \arrow[d, "Pf"] & m \arrow[d, dashed, "f'"] \\
PUPUm' \arrow[r, shift left, "\varepsilon_{PUm'}"] \arrow[r, shift right, "PU\varepsilon_{m'}"'] & PUm' \arrow[r, "\varepsilon_{m'}"] & m'
\end{tikzcd}$$

both left squares commute, so $\varepsilon_{m'} \cdot Pf$ must factor through the first coequalizer ε_m by a unique arrow f' as shown. If the functor $\overline{U}$ is applied to (2), we obtain

$$\begin{tikzcd}[column sep=large, row sep=huge]
F^R G^R F^R G^R \overline{\mathcal{U}}m \arrow[r, shift left, "\overline{\varepsilon}_{F^R G^R \overline{\mathcal{U}}m}"] \arrow[r, shift right, "F^R G^R \overline{\varepsilon}_{\overline{\mathcal{U}}m}"'] \arrow[d, "F^R G^R F^R f"'] & F^R G^R \overline{\mathcal{U}}m \arrow[r, "\overline{\varepsilon}_{\overline{\mathcal{U}}m}"] \arrow[d, "F^R f"] & \overline{\mathcal{U}}m \arrow[d, "\overline{\mathcal{U}}f'"] \\
F^R G^R F^R G^R \overline{\mathcal{U}}m' \arrow[r, shift left, "\overline{\varepsilon}_{F^R G^R \overline{\mathcal{U}}m'}"] \arrow[r, shift right, "F^R G^R \overline{\varepsilon}_{\overline{\mathcal{U}}m'}"'] & F^R G^R \overline{\mathcal{U}}m' \arrow[r, "\overline{\varepsilon}_{\overline{\mathcal{U}}m'}"] & \overline{\mathcal{U}}m'
\end{tikzcd}$$

because $\overline{\mathcal{U}}\varepsilon = \overline{\varepsilon}\,\overline{\mathcal{U}}$, where $\overline{\varepsilon}$ is the counit of the adjunction $A^R \underset{F^R}{\overset{G^R}{\rightleftarrows}} A$. But the right square commutes also for $f : \overline{\mathcal{U}}m \longrightarrow \overline{\mathcal{U}}m'$ and since $\overline{\varepsilon}_{\overline{\mathcal{U}}m}$ is the coequalizer of $\overline{\varepsilon}_{F^R G^R \overline{\mathcal{U}}m}$ and $F^R G^R \overline{\varepsilon}_{\overline{\mathcal{U}}m}$, we get $\overline{\mathcal{U}}f' = f$. Hence $\mathcal{U}$ is full and (i) holds.

If $\mathbf{A}$ has kernel pairs of split epis, then any condition of Theorem 4 is equivalent with the following one: $f : m \longrightarrow m'$ in $\mathcal{M}$, $\mathcal{U}f$ split epi implies that f is a coequalizer (see [13], Lemma 4). The equivalence (i) $\Longleftrightarrow$ (iv) is proved in [1] for the case $A = \mathbf{Ens}$. The supposition that $\mathcal{U}$ has a left adjoint is necessary.

<u>Example 1</u>. Let $P^+ : \mathbf{Ens} \longrightarrow \mathbf{Ens}$ be the covariant power set functor. Let x be an infinite set, $z \subseteq x$ an infinite subset and $\varkappa : P^+x \longrightarrow P^+x$ a constant mapping with the value z . Let $\mathcal{M}$ be a full subcategory of the category $\mathbf{Ens}(Id, Id)_{P^+}$ having one object $(x, \varkappa)$ and

$\mathcal{U}(x,\varkappa) = x$. Therefore $(M, \mathcal{U})$ is P^+-nicely embeddable into the category of unary algebras and we are going to show that M cannot be realized into any $\mathbf{Ens}^{T}$. Let C be the set of all arrows of M . Clearly $C = \{f: x \to x \,/\, P^+(f)\varkappa = \varkappa\}$. We shall use the characterization of endomorphism semigroups of T-algebras given after Theorem 1. Assume that M can be realized into some $\mathbf{Ens}^{T}$. Let $u, v \in x$, $u \in \varkappa$, $v \notin \varkappa$ and define $g: x \to x$ by $gt = t$ for $u \neq t \neq v$, $gu = v$ and $gv = u$. Since $P^+(g)\varkappa \neq \varkappa$, $g \notin C$ and by our characterization there exists $(y_f)_{f \in x^x} \in \prod_{f \in x^x} x$ such that $hy_f = y_{hf}$ for any $h \in C$ and $gy_{id} \neq y_g$. Let $u_1, u_2 \in \varkappa$ such that u, u_1, u_2 are mutually different and $k_1, k_2 : \varkappa - \{u\} \to \varkappa$ be bijections with $k_1(u_1) = u$, $k_1(u_2) = u_1$, $k_2(u_2) = u$ and $k_2(u_1) = u_2$. Let $f_i : x \to x$ be defined as follows for $i = 1, 2$: $f_i t = k_i t$ for $t \in \varkappa - \{u\}$, $f_i t = t$ for $t \in x - (\varkappa \cup \{v\})$, $f_i u = u$ and $f_i v = u_i$. Clearly $f_i, f_i g \in C$ for $i = 1, 2$. Hence $f_i g(y_{id}) = y_{f_i g} = f_i(y_g)$. Since $gy_{id} \neq y_g$, the set $\{gy_{id}, y_g\}$ has to be equal to $\{u, u_1\}$ or $\{v, u_2\}$ by the definition of f_1 and on the other hand the construction of f_2 implies that this set must be equal to $\{u, u_2\}$ or $\{v, u_1\}$. But this is a contradiction.

<u>Theorem 4'</u>. Equivalences given in Theorem 4 remain correct if we add to (i), (ii) and (iii) the condition that the occurring embeddings have a left adjoint and to (iv) and (v)

that $\mathbf{M}$ has coequalizers for all pairs $f, g : m \rightrightarrows m'$ in $\mathbf{M}$ such that $\mathcal{U}f$, $\mathcal{U}g$ have a split coequalizer in $\mathbf{A}$.

Proof: Again, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ $\Longrightarrow$ (v). By the tripleability theorem (v) $\Longrightarrow$ (i) (see [14], p.151, ex. 5). Let $H : \mathbf{M} \longrightarrow V\text{-}\mathcal{A}lg$ be an F-nice embedding having a left adjoint $L : V\text{-}\mathcal{A}lg \longrightarrow \mathbf{M}$ and $f, g : m \rightrightarrows m'$ a pair of arrows in $\mathbf{M}$ such that $\mathcal{U}f$, $\mathcal{U}g$ have a split coequalizer $h : \mathcal{U}m' \longrightarrow a$ in $\mathbf{A}$. Hence Fh is a split coequalizer of $F\mathcal{U}f$, $F\mathcal{U}g$. Since $|\ |_V$ creates split coequalizers (see [12], § 6), Hf, Hg have a coequalizer h'. L being a left adjoint preserves colimits and thus Lh' is a coequalizer of LHf and LHg. The counit $\varepsilon : LH \xrightarrow{\cdot} Id_M$ of our adjunction is an isomorphism because H is full and faithful and therefore f, g have a coequalizer $Lh' . \varepsilon_m^{-1}$. We have proved that (iii) $\Longrightarrow$ (v).

Theorem 5. Let A have countable copowers, $(\mathcal{M}, \mathcal{U})$ be structured over A and $\mathcal{U}$ have a right adjoint. Then the following conditions are equivalent:

(i) $\mathbf{M}$ is nicely embeddable into some category monadic over $\mathbf{A}$

(ii) $\mathbf{M}$ is nicely embeddable into a category $V\text{-}\mathcal{A}lg$ for some $V : \mathcal{X} \longrightarrow \mathbf{B}$

(iii) $\mathcal{U}$ has the property $(*)^{op}$ from Theorem 3

(iv) $\mathcal{U}$ reflects split equalizers.

Proof: Clearly (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv). By Theorem 3 (ii) $\Longrightarrow$ (iii). Let (iv) hold. The dual to the implication (v) $\Longrightarrow$ (i) from Theorem 4 says that $\mathbf{M}$ is realizable into a comonadic category. The condition (i) holds by Corollary 3.

§ 4. $\mathcal{A} = \mathbf{Ens}$ and full embeddings

Lemma 3. Let $(\mathcal{M}, \mathcal{U})$ be structured over $\mathcal{A}$, $\mathcal{U}$ have a left adjoint $\mathbf{P}$ and $\mathcal{N}$ be a small codense subcategory of $\mathcal{M}$ with the inclusion functor $\mathbf{K} : \mathcal{N} \longrightarrow \mathbf{M}$. Then $\mathbf{R}_U = \mathbf{R}_{UK}$.

Proof: Since $\mathcal{U}$ has a left adjoint, it preserves all right Kan extensions ([14], p. 239, Th.1), i.e. $\mathbf{Ran}_{UK} \mathcal{U}\mathbf{K} = \mathcal{U} . \mathbf{Ran}_{UK} \mathbf{K}$. Further, $\mathbf{Ran}_K \mathbf{K} = \mathbf{Id}_M$ because $\mathcal{N}$ is codense in $\mathcal{M}$. The functor $\mathbf{P}$ as a left adjoint for $\mathcal{U}$ is equal to $\mathbf{Ran}_U \mathbf{Id}_M$ (see [14], p. 245 Prop.3). Hence $\mathbf{Ran}_U \mathbf{Ran}_K \mathbf{K} = \mathbf{P}$ and therefore $\mathbf{Ran}_{UK} \mathbf{K} = \mathbf{P}$ (by Dubuc, see [14], p. 239 ex. 3). Putting all these facts together we obtain that $\mathbf{R}_U = \mathcal{U}\mathbf{P} = \mathcal{U} . \mathbf{Ran}_{UK} \mathbf{K} = \mathbf{Ran}_{UK} \mathcal{U}\mathbf{K} = \mathbf{R}_{UK}$.

TB-functors were defined in [17] as a special class of set functors $\mathbf{Ens} \longrightarrow \mathbf{Ens}$ (contravariant are admitted, too). We shall not need the precise definition of TB-functors, for our purposes it suffices to know that any hom-functor $\mathbf{Ens}(a, -)$ or $\mathbf{Ens}(-, a)$ for $a \in \mathbf{Ens}$ is a TB-functor and that the class of TB-functors is closed under compositions and all limits and colimits over small diagrams.

Lemma 4. $\mathbf{R}_U$ is a TB-functor for any small concrete category $(\mathbf{M}, \mathcal{U})$.

Proof: Denote by $\mathcal{D} : \mathcal{M}^{op} \times \mathcal{M} \longrightarrow \mathcal{E}ns^{\mathcal{E}ns}$ a functor defined by $\mathcal{D}(m', m) = \mathcal{U}m^{\mathcal{E}ns(-,\mathcal{U}m')} = \mathcal{E}ns(\mathcal{E}ns(-,\mathcal{U}m'),\mathcal{U}m)$. It was quoted in § 1 that $\mathcal{R}_{\mathcal{U}} = \mathcal{R}an_{\mathcal{U}}\mathcal{U} = \int_m \mathcal{D}(m, m)$. Any functor $\mathcal{D}(m', m)$ is, as a composition of hom-functors, a TB-functor and further, the subdivision category $\mathcal{M}^{\S}$ is small because $\mathcal{M}$ is small. Hence $\mathcal{R}_{\mathcal{U}}$ is a small limit of TB-functors and thus is a TB-functor itself.

Dually a left Kan extension $\mathcal{L}$ of $\mathcal{U}$ along $\mathcal{U}$ is a TB-functor because $\mathcal{L}$ is a small colimit of functors $\mathcal{E}ns(\mathcal{U}m', -) \times \mathcal{U}m$.

Supposing $(\mathcal{M})$, many equational categories without a rank can be strongly embedded into some $\mathcal{Ol}(\Delta)$, e.g. complete lattices, complete Boolean algebras, compact Hausdorff spaces and complete Boolean algebras with closure operation (see [19]).

Theorem 6. Let $(\mathcal{M})$ hold. If T is a TB-functor, then the varietal category $\mathcal{E}ns^{T}$ is strongly embeddable into a category $\mathcal{Ol}(\Delta)$ for some type Δ . It holds whenever $T = \mathcal{R}_{\mathcal{U}}$ for a $\mathcal{U} : \mathcal{M} \longrightarrow \mathcal{E}ns$ with a small $\mathcal{M}$ and particularly if $\mathcal{E}ns^{T}$ has a small codense subcategory.

Proof: $\mathcal{E}ns^{T}$ is a full subcategory of the category $\mathcal{E}ns(T, Id_{\mathcal{E}ns})$ which is strongly embeddable into some $\mathcal{Ol}(\Delta)$ for a TB-functor T by [19] 3.11. The rest follows from Lemmas 4 and 3.

An example of a varietal category with a small codense subcategory is the category of compact Hausdorff spaces in which the unit interval $[0,1]$ forms a codense subcategory.

Problem 1: Can any equational (varietal) category be strongly embedded into some $\mathcal{U}(\Delta)$ under (M) ?

The full embeddability of some equational categories without a rank into $\mathcal{U}(\Delta)$ implies (M), e.g. of compact Hausdorff spaces, complete Boolean algebras, compact Hausdorff Boolean algebras and of compact Hausdorff abelian groups (see [9], [10]). We may ask whether there exists an equational (varietal) category without a rank which can be fully embedded into some $\mathcal{U}(\Delta)$ under $non(M)$.

Now, we turn our attention to full embeddings of concrete categories into equational categories. Kučera quotes in [9] the result of Trnková that any concrete category can be fully embedded into the category of topological T_1-spaces and continuous open mappings. By [8] the category of topological spaces and continuous open mappings is dual to the category of complete Boolean algebras with closure operation, which is an equational category.

Hence any concrete category can be fully embedded into some equational category.

Problem 2: Is any concrete category fully embeddable into some varietal category (without (M))?

Problem 3: Let A be an arbitrary category. To study full embeddings of categories structured over A into monadic categories over A or into categories $V\text{-}\mathcal{Alg}$ for $V: X \longrightarrow A$.

As a small contribution to the last problem we shall give the following results.

<u>Theorem 7</u>. Let $\mathcal{A}$ be a category, $(\mathcal{E}ns, W)$ be a structured category over $\mathcal{A}$ which is realizable into some monadic category over $\mathcal{A}$ and W have a faithful left adjoint $P: \mathcal{A} \longrightarrow \mathcal{E}ns$. Then any category $(\mathcal{M}, \mathcal{U})$ structured over $\mathcal{A}$ having a small dense subcategory $\mathcal{N} \subseteq \mathcal{M}$ can be fully embedded into some category monadic over $\mathcal{A}$.

Under (M), it holds for any $(\mathcal{M}, \mathcal{U})$.

<u>Proof</u> : $(\mathcal{M}, P\mathcal{U})$ is concrete and thus supposing (M) it can be fully embedded into some $\mathfrak{A}(\Delta)$ (if $\mathcal{M}$ has a small dense subcategory it holds without (M) by [7]). Let $\mathcal{U}' : \mathfrak{A}(\Delta) \longrightarrow \mathcal{E}ns$ be the forgetful functor. Thus $(\mathfrak{A}(\Delta), W\mathcal{U}')$ is structured over $\mathcal{A}$ and $W\mathcal{U}'$ has a left adjoint. If we show that $W\mathcal{U}'$ has the property $(*)$ from Theorem 3, then Theorem 7 will follow from Theorem 4. But it holds by the facts that W and $\mathcal{U}'$ have this property and that a faithful functor reflects epis.

An example of a category $\mathcal{A}$ from this theorem is any concrete category $(\mathcal{A}, P)$ such that P has a right adjoint W which is a full embedding $W: \mathcal{E}ns \longrightarrow \mathcal{A}$. For instance, such a category $\mathcal{A}$ is the category of graphs or the category of topological spaces (and continuous mappings).

Finally, we shall give an example of a category $\mathcal{A}$ and a small category structured over $\mathcal{A}$ which cannot be fully embedded into any category $V\text{-}\mathcal{A}lg$ for $V: \mathcal{X} \longrightarrow \mathcal{A}$.

Example 2. Let A be a category with the only one object a and with arrows $f_n : a \longrightarrow a$ for any integer n. The composition is defined by $f_k \cdot f_m = f_{k+m}$, i.e. $f_0 = id_a$. Therefore any arrow in A is an isomorphism. Categories structured over A having the only one object are in 1-1 correspondence with subsemigroups of the additive group of integers. Let M be a subcategory of A having arrows f_n for $n > 0$ and $U : M \longrightarrow A$ the inclusion. Thus (M, U) is structured over A and $M(a, a)$ is the semigroup generated by f_1. Let $H : M \longrightarrow V\text{-}Alg$ be a full embedding, where $V : X \longrightarrow A$ is a functor. Hence the semigroup of endomorphisms of the V-algebra Ha is generated by Hf_1. Let $||_V H(f_1) = f_m$. Define $Ff_n = f_{mn}$. Then $F : A \longrightarrow A$ is a functor because $F(f_k \cdot f_n) = Ff_{k+n} = f_{m(k+n)} = f_{mk+mn} = Ff_k \cdot Ff_n$. Let f_n be an arrow of M. It holds

$$FUf_n = f_{mn} = \underbrace{f_m \ldots f_m}_{n \times} = \underbrace{||_V Hf_1 \ldots ||_V Hf_1}_{n \times} = ||_V H(\underbrace{f_1 \ldots f_1}_{n \times}) = ||_V Hf_n .$$

Hence H is an F-strong embedding. But, by Theorem 3 M is not strongly embeddable into any category $V\text{-}Alg$ because U does not reflect isomorphisms.

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