Rostislav Zezula

Weak and strong convergence of projection methods in nonreflexive Banach spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 15 (1974), No. 1, 29--47

Persistent URL: `http://dml.cz/dmlcz/105531`

Commentationes Mathematicae Universitatis Carolinae

15,1 (1974)

WEAK AND STRONG CONVERGENCE OF PROJECTION METHODS IN NONREFLEXIVE BANACH SPACES

Rostislav ZEZULA, Praha

Abstract: In this paper, two theorems are proved, concerning weak (or strong resp.) convergence of projection methods for solving the operator equation $Ax = f$ in a Banach space X (with $A: X \to X$ in general nonlinear and X in general nonreflexive). Further, there are proved three lemmas enabling us to verify the assumptions in these theorems.

Key words: Nonlinear mappings in nonreflexive normed linear spaces, projection methods for solving operator equations, weak solutions, strong solutions.

AMS: Primary 47H15 Ref. Ž. 7.978, 8.31
Secondary 65J05

Introduction. W.V. Petryshyn, F.E. Browder and other authors (see references in [4]) have studied, by help of functional analytic methods, the convergence problem for projection methods applied to operators (in general nonlinear) in Banach spaces having certain approximation and compacticity properties, and also the use of projection methods for a constructive proof of existence and unicity of solutions to operator equations in these spaces.

This paper, starting from the above mentioned results contained in [4] and from the author's papers [1],[2],[3] gives sufficient conditions for the existence, unicity and approximative construction of solutions to operator equations in Banach spaces for the case that we know further informations concerning the operators in these equations. Similarly as in [1],[2] the assumptions imposed to the operators are stronger than in [4], but the assumptions concerning the space in which the operator equation is considered are somewhat weakened.

§ 1

Notations. Let us denote X a real or complex Banach space (in general nonreflexive), X^* the corresponding dual or antidual space, $A: X \longrightarrow X$ an operator (in general nonlinear) mapping X into X, and $\langle v, u \rangle$, the value of the functional $v \in X^*$ applied to the element $u \in X$. Let $m, i \in \mathbb{N}$ where $\mathbb{N}$ denotes the index set of positive integers.

We shall study the problem of existence, unicity and approximative construction of the solution $x \in X$ to the operator equation in the space X [4],[2]

$$(1.1) \qquad Ax = f\,, \quad A: X \longrightarrow X\,, \quad f \in X\,.$$

It is clear that, for obtaining an affirmative answer, suitable assumptions must be made concerning the space X and the operator A.

<u>Definition 1.1</u> [4]: We say that the space X has the property $(\pi)_C$, iff $\exists$ a sequence $\{X_m\}$ of finite dimensional subspaces $X_m \subset\subset X$, $dim\, X_m = m$, a sequence

$\{P_n\}$ of linear projections $P_n = P_n^2$ defined on X and a constant $C > 0$ such that we have

$$(1.2) \qquad P_n X = X_n \,, \; X_n \subset X_{n+1} \quad \text{for } \forall n \in \mathbb{N}, \; \overline{\bigcup_{n=1}^{\infty} X_n} = X$$

$$(1.3) \qquad \|P_n\| \le C \quad \text{for} \quad \forall n \in \mathbb{N}, \; P_n P_j = P_j \quad \text{for } n \ge j \in \mathbb{N} \,.$$

Now let $\{A_n : X \to X\}$, $n = 1, 2, \dots$ be a sequence of approximative operators and let for each $i \in \mathbb{N}$ there exist an index $n_i \in \mathbb{N}$ and an element $f_i \in X$ such that $f_i \in A_{n_i}(X)$. Let us consider the approximative equation

$$(1.4) \qquad A_{n_i} x = f_i \quad .$$

Then the following definition is useful:

<u>Definition 1.2</u> [4] : We say that the equation (1.1) is projectionally strongly solvable or PS-solvable (projectionally weakly solvable or PW-solvable, resp.), if the following three propositions are valid:

i) the equation (1.1) has the unique solution $x \in X$, $Ax = f$ for $\forall f \in X$

ii) $\exists N \in \mathbb{N}$ such that for $\forall n_i \ge N$ and for $\forall f_i \in A_{n_i}(X$ the approximative equation (1.4) has the unique solution

$$x_i \in A_{n_i}(X) \,, \quad A_{n_i} x_i = f_i$$

iii) $f_i \to f \Longrightarrow x_i \to x$, $Ax = f$ (PS-solvability)

$f_i \to f \Longrightarrow x_i \rightharpoonup x$, $\langle v, Ax \rangle = \langle v, f \rangle$ for $\forall v \in X_0^* \subset X^*$ (PW-solvability) resp.

In Petryshyn's paper [4], there is solved the following

<u>Problem A</u>: To find sufficient conditions for PS-solvability (PW-solvability resp.) of the operator equation (1.1) in <u>reflexive</u> Banach spaces $\mathcal{X}$.

With respect to the fact that, in practice, one often knows further informations concerning the operators A_m, A resp., it seems to be useful to formulate the following two problems:

<u>Problem B</u> ([1],[2]) : We know that the approximative equation (1.4) has the unique solution $x_i \in A_{m_i}(\mathcal{X})$ for $\forall f_i \in A_{m_i}(\mathcal{X})$ and for $\forall m_i \geq N \in \mathbb{N}$. We have to seek for sufficient conditions ensuring that $f_i \longrightarrow f \in \mathcal{X} \Longrightarrow x_i$ converges to $x \in \mathcal{X}$ (weakly or strongly) and that this x is the unique solution (weak or strong) to the operator equation (1.1).

<u>Problem C</u> ([2]): Let $\{B_m : \mathcal{X} \longrightarrow \mathcal{X}\}$ be a sequence of operators in $\mathcal{X}$. We know that the equation (1.1) has a unique (weak or strong) solution $x \in \mathcal{X}$ for $\forall f \in A(\mathcal{X})$. We have to seek for sufficient conditions ensuring that for $A_m = A - B_m$ there exists a unique solution $x_m \in A_m(\mathcal{X})$ to the approximate operator equation (1.4) for $\forall f_m \in A_m(\mathcal{X})$ and $\forall m \geq N \in \mathbb{N}$, and that $f_m \to f \Longrightarrow x_m$ converges to x (weakly or strongly).

§ 2.

In this paragraph, we shall give a solution to Problem B.

<u>Theorem 2.1</u>. Let $Ax = f$ be a given operator equation in the real or complex Banach space $\mathcal{X}$ (in general nonreflexive), where $f \in \mathcal{X}$ and $A : \mathcal{X} \longrightarrow \mathcal{X}$ is a weakly continu-

ous on $X_0^* \subset X^*$ (demicontinuous resp.) operator, in general nonlinear (i.e., we have $x_n \in X, x_n \rightharpoonup x \in X, w \in X_0^* \subset X^*$ $\Longrightarrow \langle w, Ax_n \rangle \longrightarrow \langle w, Ax \rangle$ (or $x_n \to x \Longrightarrow Ax_n \rightharpoonup Ax$ resp.)).

Let the operator A and space X fulfil the following assumptions:

(i) $\exists$ a sequence $\{A_n\}$ of operators $A_n : X \longrightarrow X$ such that their restriction $\widetilde{A}_n = A_n/_{A_n(X)}$ to $A_n(X)$ is bijective with $\widetilde{A}_n^{-1} : A_n(X) \longrightarrow A_n(X)$ as inverse operator and such that the relation

$$(2.1) \qquad \overline{\bigcup_{n=1}^{\infty} A_n(X)} = X$$

holds.

(ii) If for $\forall i \in \mathbb{N} \exists m_i \in \mathbb{N}$ such that $f_i \in A_{m_i}(X)$, then we have

$$(2.2) \quad f_i \in A_{m_i}(X), f_i \to f \Longrightarrow (A\widetilde{A}_{m_i}^{-1} - I) f_i \rightharpoonup 0 \quad \text{for } i \to \infty$$

(where I denotes the identity operator in $A_{m_i}(X)$ and $\longrightarrow$ or $\rightharpoonup$ resp. stands for the strong or weak convergence in X resp.).

(iii) $\exists$ a weakly continuous (or strongly continuous resp.) operator $V : X \longrightarrow X$ for which the relation (of left-hand side inverse)

$$(2.3) \qquad x \in X \Longrightarrow V(Ax) = x$$

is fulfilled.

Then the following assertions are valid:

$\forall f \in X \exists \{f_i \in A_{m_i}(X)\}, f_i \longrightarrow f$ such that we have

1) $f \in X$, $f_i \in A_{m_i}(X) \longrightarrow f \Longrightarrow x_i = \tilde{A}^{-1}_{m_i} f_i \rightharpoonup Vf$

(or $f \in X$, $f_i \in A_{m_i}(X) \longrightarrow f \Longrightarrow x_i = \tilde{A}^{-1}_{m_i} f_i \longrightarrow Vf$ resp.).

2) The operator equation $Ax = f$ has for $\forall f \in X$ the unique weak solution $x = Vf$, i.e., we have

(2.4) $\quad w \in X_0^* \subset X^* \Longrightarrow \langle w, AVf \rangle = \langle w, f \rangle$

(or the equation $Ax = f$ has for $\forall f \in X$ the unique strong solution $x = Vf$, i.e., we have $A(Vf) = f$ resp.).

3) The relation

(2.4 a) $$(\tilde{A}^{-1}_{m_i} - V) f_i \rightharpoonup 0$$

(or

(2.4b) $\quad (\tilde{A}^{-1}_{m_i} - V) f_i \longrightarrow 0 \quad$ resp.)

is valid for $i \longrightarrow \infty$.

<u>Proof</u>: (<u>Existence of a solution</u>) From (2.1) it follows that for $\forall f \in X \;\exists$ a sequence $\{f_i\} \subset \bigcup_{m=1}^{\infty} A_m(X)$, so that $\exists$ a sequence $\{m_i\} \subset N$ of indices m_i such that we have

$$f_i \in A_{m_i}(X)$$

In virtue of the assumption (i), there exists also a sequence of elements $x_i = \tilde{A}^{-1}_{m_i} f_i$ such that

$$A_{m_i} x_i = f_i \rightarrow f \quad .$$

Now, we shall prove, using (2.2), the following weak convergence in the space X :

(2.5) $$A x_i \rightharpoonup f \quad .$$

We have, clearly, for $i \to \infty$

(2.5a) $$v \in X^* \Longrightarrow \langle v, Ax_i - f \rangle = \langle v, (A\tilde{A}_{m_i}^{-1} - I) f_i \rangle + \\ + \langle v, f_i - f \rangle \longrightarrow 0$$

because the first term on the last right side [illegible]ges to zero in virtue of the assumption (ii) and the secon[illegible] goes to zero by the continuity of $v \in X^*$.

Using the weak continuity of the operator V assumed in (iii), we obtain from (2.5)

(2.6) $$Ax_i \rightharpoonup f \Longrightarrow VAx_i = x_i \rightharpoonup Vf \quad .$$

Now, let us take

(2.7) $$w \in X_0^* \subset X^*$$

so that we have clearly, by (2.3),

(2.8) $$\langle w, f_i \rangle = \langle w, A_{m_i} x_i \rangle = \langle w, Ax_i \rangle - \\ - \langle w, (A - A_{m_i}) x_i \rangle = \langle w, A(VAx_i \rangle - \langle w, (A - A_{m_i}) \tilde{A}_{m_i}^{-1} f_i \rangle$$

so that we obtain for $i \to \infty$, using (2.6), (2.2) and the assumed weak continuity of the operator A on $X_0^* \subset X^*$

(2.9) $$w \in X_0^* \subset X^*, f_i \to f, x_i \rightharpoonup Vf \Longrightarrow \langle w, AVf \rangle = \langle w, f \rangle$$

i.e., the element $Vf \in X$ is for $Vf \in X$ a weak solution of the operator equation (1.1).

(Or, if we use the assumptions that the operator V is strongly continuous, i.e., $x_i \rightharpoonup y \Longrightarrow Vx_i \longrightarrow Vy$, and that the operator A is demicontinuous, we have

$$(2.10) \qquad Ax_i \rightharpoonup f \Longrightarrow x_i = VAx_i \longrightarrow Vf$$

and therefore also

$$(2.11) \qquad Ax_i \rightharpoonup AVf \ .$$

Because the weak limit of Ax_i in X is unique, it follows from (2.10) and (2.11) that we have

$$(2.12) \qquad AVf = f$$

i.e, the element $Vf \in X$ is a solution of $Ax = f$ for $\forall f \in X$, resp.)

Assertion 3) follows in both the weak or strong version resp. from the relation

$$(2.13) \quad (\tilde{A}^{-1}_{m_i} - V)f_i = (\tilde{A}^{-1}_{m_i} f_i - Vf) + (Vf - Vf_i)$$

in virtue of the assertion 1) and Assumption (iii).

(<u>Unicity of the solution</u>): If there would exist $x', x'' \in X$ such that $Ax' = f$, $Ax'' = f$ then we would have by (2.3)

$$(2.14) \quad x' - x'' = (VAx' - VAx'') = (Vf - Vf) = \emptyset \ . \qquad \text{Q.E.D.}$$

<u>Remark 2.1</u>. Clearly, the sequence $\{f_i \in A_{m_i}(X)\}$, $f_i \longrightarrow f \in X$ may be also stationary, this case being trivial from the point of view of projection methods. But for this trivial case, too, it follows from (2.2) that we have for all sufficiently great indices i the relation $f = f_i = A_{m_i}x_i = Ax_i$, so that, by (2.3), $\tilde{A}^{-1}_{m_i} f = x_i = VAx_i = Vf$ is the unique solution to $Ax = f$, and the relation $Ax \rightharpoonup f$ is fulfilled trivially.

Remark 2.2. Some applications of Theorem 2.1 are given in [1],[5],[6].

§ 3

In this paragraph, we shall give a solution to the problem C.

Theorem 3.1. Let $Ax = f$ be a given operator equation in the real or complex Banach space X (in general nonreflexive), where $f \in X$ and $A : X \longrightarrow X$ is a weakly continuous (continuous resp.) operator, in general nonlinear.

Let the following assumptions be fulfilled:

(i) $\exists$ a continuous monotonically increasing real function $\alpha(\kappa)$ defined for $\kappa \geq 0$ such that

(3.1) $$\alpha(0) = 0 \ ,$$

(ii) $\exists$ an operator $T: X^* \longrightarrow X^*$ such that we have

(3.2) $$x, y \in X, \ v \in X^* \Longrightarrow |\langle Tv, Ax - Ay \rangle| \geq \alpha(|\langle v, x - y \rangle|)$$

so that the operator A is injective

(or

(3.2a) $$x, y \in X \Longrightarrow \|Ax - Ay\| \geq \alpha(\|x - y\|)$$

so that the operator A is injective, resp.)

(iii) $\exists$ a sequence of operators $\{B_m\}, B_m : X \longrightarrow X$ and $\exists Y \subset X$ such that we have: Y weakly closed, $Y \supset A(X)$, $(A - B_m)^2(X) = (A - B_m)(X) \subset Y$ and

(3.3) $$x \in X \Longrightarrow B_m x \rightharpoonup \emptyset \quad \text{for} \quad m \rightarrow \infty$$

(or

(3.3a) $$x \in X \Longrightarrow B_m x \longrightarrow \emptyset \quad \text{for} \quad m \rightarrow \infty \ , \text{ resp.})$$

(iv) $\exists$ a constant $\chi \in (0, 1)$ such that we have for the above mentioned operator $T : X^* \longrightarrow X^*$ and for $\forall m \in \mathbb{N}$

(3.4) $x, y \in Y, v \in X^* \Longrightarrow |\langle Tv, B_m x - B_m y \rangle| \leq K\alpha(|\langle v, x - y \rangle|)$

(or $\exists$ a constant $K \in (0,1)$ such that we have for $\forall m \in \mathbb{N}$

(3.4a) $x, y \in Y \Longrightarrow \|B_m x - B_m y\| \leq K\alpha(\|x - y\|)$ resp.)

Then for the sequence of operators

(3.5) $\qquad A_m = A - B_m , \qquad A_m : X \longrightarrow X$

the following assertions are valid:

1) For $\forall m \in \mathbb{N}$, the operators A_m are injective (on Y).

2) We have

(3.6) $\quad w \in X_0^* = T(X^*),\ f_m \in A_m(X),\ f_m \longrightarrow f \in A(Y) \Longrightarrow x_m =$

$$= A_m^{-1} f_m \longrightarrow x = A^{-1} f$$

and $f_m \longrightarrow f \in Y, x_m \longrightarrow x \Longrightarrow \langle w, Ax \rangle = \langle w, f \rangle$, $x \in Y$ unique (or

(3.6a) $f_m \in A_m(X),\ f_m \longrightarrow f \in A(Y) \Longrightarrow x_m = A_m^{-1} f_m \longrightarrow x = A^{-1} f$

and $f_m \longrightarrow f \in Y,\ x_m \longrightarrow x \Longrightarrow Ax = f \Longleftrightarrow f \in A(Y), x$ unique, resp.)

3) We have the following error estimation by help of the residuum $r_m = f - Ax_m$:

$f \in Y,\ v \in X^*,\ w = Tv,\ \langle w, Ax \rangle = \langle w, f \rangle \Longrightarrow |\langle Tv, f - Ax_m \rangle| =$

$$= |\langle Tv, Ax - Ax_m \rangle| \geq \alpha(\langle v, x - x_m \rangle)$$

and therefore, denoting $\beta = \beta(r),\ r \geq 0$ the inverse function to $\alpha(r)$

(3.6b) $\quad 0 \longleftarrow \beta(|\langle Tv, f - Ax_m \rangle|) \geq |\langle v, x - x_m \rangle|$

(which represents a componentwise estimation for the error $x - x_m$ of the weak solution $x \in Y$ to the equation $Ax = f$, (or

$$x_m \to x,\ Ax = f \Longrightarrow \|\varkappa_m\| = \|f - Ax_m\| = \|Ax - Ax_m\| \geq \alpha(\|x - x_m\|)$$

and therefore

(3.6c) $\quad 0 \longleftarrow \beta(\|f - Ax_m\|) \geq \|x - x_m\| \qquad$ resp.)

<u>Proof</u> (<u>Injectivity of</u> A_m). We have, following (3.2) and (3.4),

(3.7) $\quad v \in X^*,\ x, y \in Y,\ A_m x = A_m y \Longrightarrow 0 = |\langle Tv, A_m x - A_m y\rangle| =$

$$= |\langle Tv, Ax - Ay\rangle - \langle Tv, B_m x - B_m y\rangle| \geq \alpha(|\langle v, x - y\rangle|) -$$

$$- K\,\alpha(|\langle v, x - y\rangle|) = (1 - K)\,\alpha(|\langle v, x - y\rangle|)$$

and therefore, because $(1 - K) > 0$ by the assumption (iv), it follows from (3.7) by help of the properties of the function $\alpha(\varkappa)$ assumed in (i), that we have

$$|\langle v, x - y\rangle| = 0 \quad \text{for } \forall v \in X^* \Longrightarrow x = y$$

so that A_m is injective for $\forall m \in \mathbb{N}$ on Y (or we have, following (3.2a) and (3.4a),

(3.7a) $\quad x, y \in Y,\ A_m x = A_m y \Longrightarrow 0 = \|A_m x - A_m y\| \geq$

$$\geq \|Ax - Ay\| - \|B_m x - B_m y\| \geq$$

$$\geq (1 - K)\,\alpha(\|x - y\|)$$

and therefore, by the assumption (i), it follows from (3.7a) that

$$\|x - y\| = 0 \Longrightarrow x = y$$

so that again A_m is injective for $\forall m \in \mathbb{N}$ on Y.

(<u>Construction of the solution</u>) Let $f_m \in A_m(X)$, $f_m \to$ $\to f \in A(Y)$. For the unique solutions $x_m \in A(X), x \in Y$ resp.

to the operator equations $A_n x_n = f_n$, $Ax = f$ resp., the relation

$$(3.8) \quad f - f_n = Ax - A_n x_n = (A - A_n)x + (A_n x - A_n x_n) = \\ = B_n x + (A_n x - A_n x_n)$$

must be valid following (3.5), from which we obtain with help of the inequality (3.7) and of (iii)

$$(3.9) \quad v \in X^* \Longrightarrow |\langle Tv, f - f_n \rangle| = |\langle Tv, B_n x \rangle + \langle Tv, A_n x - \\ - A_n x_n \rangle| \geq (1-K)\alpha(|\langle v, x - x_n \rangle|) - |\langle Tv, B_n x \rangle|$$

so that, for $n \to \infty$, we obtain by help of (3.3), (3.9)

$$(3.10) \quad 0 \leq (1-K)\alpha(|\langle v, x - x_n \rangle|) \leq |\langle Tv, f - f_n \rangle| + \\ + |\langle Tv, B_n x \rangle| \longrightarrow 0$$

for $\forall v \in X^*$ because $B_n x \rightharpoonup 0$, $f_n \to f$ and $Tv \in X^*$ is a continuous linear functional and therefore, in virtue of the assumption (i), we have for $f_n \in A_n(X)$, $f_n \to f \in A(Y)$

$$(3.11) \quad \langle v, x - x_n \rangle \longrightarrow 0 \quad \text{for } \forall v \in X^* \quad \text{i.e.,} \quad x_n = \\ = A_n^{-1} f_n \rightharpoonup x = A^{-1} f \ .$$

Now we shall prove that also for $f_n \to f \in Y$ the element x which is the weak limit of $x_n = A_n^{-1} f_n$ (if it exists), is a weak solution to the operator equation (1.1) (the case of a generalized solution). Clearly, we have, in virtue of (3.5), (3.4) for $\forall w = Tv \in X_0^* = T(X^*)$ and for $f_n \to f \in Y$, $f_n \in A_n(X)$ the relation (because Y is weakly closed by (iii))

$$(3.12)\quad \langle w, f_m\rangle = \langle w, A_m x_m\rangle = \langle w, Ax_m\rangle -$$

$$- \langle w, (A-A_m)x_m\rangle = \langle w, Ax_m\rangle - \langle w, B_m x_m\rangle =$$

$$= \langle w, Ax_m\rangle - \langle w, B_m x\rangle + \langle Tv, B_m x - B_m x_m\rangle$$

from which it follows, using (3.4) and (iii)

$$(3.13)\quad 0 \le |\langle w, f_m - Ax_m\rangle| \le |\langle Tv, B_m x_m\rangle - \langle w, B_m x\rangle| \le$$

$$\le K\alpha(|\langle v, x - x_m\rangle|) + |\langle w, B_m x\rangle|$$

But the operator A is, by assumption, weakly continuous, so that it follows from (3.11)

$$(3.14)\quad x_m \rightharpoonup x \Longrightarrow Ax_m \rightharpoonup Ax,\ f_m \to f \in X \Longrightarrow f_m - Ax_m \rightharpoonup f - Ax\ .$$

Clearly, we obtain from (3.13), using (3.11), (i), (3.3) and (3.14), for $m \to \infty$

$$(3.15)\quad 0 \le \lim_{m\to\infty} |\langle w, f_m - Ax_m\rangle| = |\langle w, f - Ax\rangle| \le$$

$$\le \lim_{m\to\infty} \{K\alpha(|\langle v, x - x_m\rangle|) + |\langle w, B_m x\rangle|\} = 0$$

and therefore

$$(3.16)\quad \langle w, f - Ax\rangle = 0 \qquad \text{for } \forall w \in X_0^* = T(X^*)$$

i.e., if $f_m \to f \in Y$, $x_m \rightharpoonup x$, then x is a generalized weak solution to (1.1).

(Or, if we suppose the continuity of the operator A and the validity of the relations (3.2a), (3.3a), (3.4a), we obtain, estimating the norms in the relation (3.8) by help of the inequality (3.7a) and (iii), for $f_m \in A_m(X)$, $f_m \to f \in A(Y)$

$$(3.17)\quad \|f - f_m\| \ge \|B_m x\| + (1-K)\,\alpha(\|x - x_m\|)\ .$$

Because $f_m \to f$, it follows from the inequality (3.17), using (3,3a), (iv) and (i) and letting $m \to \infty$

$$(3.18)\quad \|x - x_m\| \to 0\,,\ \text{i.e. } x_m = A_m^{-1} f_m \longrightarrow x = A^{-1} f \in X\ .$$

Because the operator A is continuous by assumption, we have

$$(3.19)\qquad x_m \longrightarrow x = A^{-1} f \Longrightarrow A x_m \longrightarrow A x = f\ .)$$

Now, we shall prove that for $f_m \longrightarrow f \in Y$, the element $x \in X$ which is the strong limit of $x_m = A_m^{-1} f_m \in A_m(X)$ (if it exists), is a solution to the operator equation (1.1), i.e., $Ax = f$, so that $f \in A(X)$ (no generalized solutions). Clearly, we have for $f_m \to f \in Y$, $x_m = A_m^{-1} f_m \longrightarrow x \in Y$ by (iii) and (3.5)

$$(3.20)\qquad f_m = A_m x_m = A x_m - (A - A_m) x_m = A x_m - B_m x_m = \\ = A x_m - B_m x + (B_m x - B_m x_m)$$

so that we obtain by estimating the norms in (3.20) by help of the inequality (3.4a)

$$(3.21)\quad 0 \le \|f_m - A x_m\| \le \|B_m x\| + K\alpha(\|x - x_m\|)\ .$$

Because A is supposed to be continuous and $f_m \longrightarrow f \in X$, $x_m \longrightarrow x$ we have

$$(3.22)\qquad f_m - A x_m \longrightarrow f - A x$$

and therefore, letting $m \longrightarrow \infty$ in (3.21), we obtain by help of the assumptions (3.3a) and (i)

$$(3.23)\quad 0 \le \|f - A x\| \le 0 \Longleftrightarrow A x = f\,,\quad \text{i.e. } f \in A(X) = Y.$$

(<u>Error estimation</u>) The componentwise error estimation (3.6b) follows immediately from the inequality (3.2) in the assumption (ii) of Theorem 3.1.

Similarly, the norm error estimation (3.6c) follows immediately from the inequality (3.2a) in the injectivity assumption (ii) of Theorem 3.1.

(<u>Unicity of the solution</u>) Unicity of the solution to the equation (1.1) for $f \in A(\mathcal{X})$ follows from the fact that A is assumed to be injective. For weak solutions and $f \in \mathcal{X} \setminus A(\mathcal{X})$ (generalized solutions) we can prove the uniqueness of the weak solution to (1.1) in the following way: let there exist for $f \in \mathcal{X} \setminus A(\mathcal{X})$ two elements $x' \in \mathcal{X}$, $x'' \in \mathcal{X}$ such that $\langle w, Ax' \rangle = \langle w, f \rangle$ and $\langle w, Ax'' \rangle = \langle w, f \rangle$ for $\forall w = Tv \in \mathcal{X}_0^* = T(\mathcal{X}^*)$. Then we have, following (3.2),

$$(3.24) \quad 0 = |\langle w, Ax' - Ax'' \rangle| = |\langle Tv, Ax' - Ax'' \rangle| \geq \alpha(|\langle v, x' - x'' \rangle|)$$

and therefore, using the assumption (i), we obtain

$$(3.25) \qquad \langle v, x' - x'' \rangle = 0$$

for $\forall v \in \mathcal{X}$, so that $x' = x''$. Q.E.D.

§ 4.

In this last paragraph, we shall prove three simple lemmas which will enable us to verify the most important assumptions in the theorem 2.1.

The assumption (ii) in Theorem 2.1 can be verified by help of the following

<u>Lemma 4.1</u>. (i) Let the sequence $\{C_n\}$ of operators

$C_m : \mathcal{X} \longrightarrow \mathcal{X}$ be so that their restrictions $\widetilde{C}_m = C_m|_{C_m(\mathcal{X})}$ to $C_m(\mathcal{X})$ are bijective having the inverse operator $(\widetilde{C}_m)^{-1}$.

(ii) Let $\exists$ an injective operator $C : \mathcal{X} \longrightarrow \mathcal{X}$ having the following properties:

$$(4.1) \qquad x \in \mathcal{X} \Longrightarrow C_m x \longrightarrow Cx$$

$$(4.1a) \qquad C(\mathcal{X}) \supset C_m(\mathcal{X}) \quad \text{for } \forall m \in \mathbb{N} \ .$$

(iii) For $\forall m \in \mathbb{N}$, the operators $C(\widetilde{C}_m)^{-1}$ are Lipschitz operators with the Lipschitz constants α, $0 < \alpha_m \leq \alpha < \infty$, so that we have

$$(4.2) \qquad x', x'' \in C_m(\mathcal{X}),\ m \in \mathbb{N} \Longrightarrow \| C(\widetilde{C}_m)^{-1}x' - C(\widetilde{C}_m)^{-1}x''\| \leq \leq \alpha \|x' - x''\| \ .$$

Then the relation

$$(4.3) \quad z_i \in C_{m_i}(\mathcal{X}),\ z_i \longrightarrow z \in C(\mathcal{X}) \Longrightarrow \langle v, (C - C_{m_i})(\widetilde{C}_{m_i})^{-1} z_i \rangle \rightarrow 0$$

is valid for $\forall v \in \mathcal{X}^*$.

<u>Proof</u>. In virtue of the injectivity assumption (ii), there exists the inverse operator $C^{-1} : C(\mathcal{X}) \longrightarrow \mathcal{X}$ so that we have

$$(4.4) \qquad z \in C(\mathcal{X}) \Longrightarrow \exists x = C^{-1} z \in \mathcal{X} \ .$$

For $z_i \in C_{m_i}(\mathcal{X}) \subset C(\mathcal{X}), z_i \rightarrow z \in C(\mathcal{X}), x_i = (\widetilde{C}_{m_i})^{-1} z_i, x = C^{-1} z, x_i' = C^{-1} z_i$ we have, using the triangle inequality and the boundedness of $v \in \mathcal{X}^*$,

$$(4.4a) \quad |\langle v, (C - C_{m_i})(\widetilde{C}_{m_i})^{-1} z_i \rangle| \leq |\langle v, C(\widetilde{C}_{m_i})^{-1} z_i - z\rangle| + + |\langle v, z - z_i\rangle| \leq \|v\| \cdot [\|C(\widetilde{C}_{m_i})^{-1} z_i - z\| + \|z - z_i\|] \longrightarrow 0$$

for $m \to \infty$, because we have $\| z - z_i \| \to 0$ by the continuity of the norm and further

$$(4.4b) \quad \| C(\tilde{C}_{m_i})^{-1} z_i - z \| = \| C(\tilde{C}_{m_i})^{-1} C x_i' - C(\tilde{C}_{m_i})^{-1} C_{m_i} x \| \le$$

$$\le \alpha \| C x_i' - C_{m_i} x \| \le \alpha \| C x_i' - C x \| + \alpha \| C x - C_{m_i} x \| =$$

$$= \alpha \| z_i - z \| + \alpha \| C x - C_{m_i} x \| \to 0$$

for $i \to \infty$, in virtue of the assumptions (4.1), (4.2).

Q.E.D.

The assumption concerning the injectivity of the operators A_m in the theorem 2.1 can be verified using the following

Lemma 4.2. Let $\alpha_m(\varkappa)$ be a sequence of continuous monotonically increasing real functions defined for $\varkappa \ge 0$ and such that $\alpha_m(0) = 0$. Let us suppose that, for $\forall m \ge N \in \mathbb{N}$, the operators $A_m : X \to X$ fulfil the condition

$$(4.5) \quad x', x'' \in X \Longrightarrow \| A_m x' - A_m x'' \| \ge \alpha_m (\| x' - x'' \|) .$$

Then, for $\forall m \ge N \in \mathbb{N}$, the operators A_m are injective.

Proof: By contradiction.

If there would exist $x', x'' \in X$, such that $A_m x' = A_m x''$ and $x' \neq x''$ for some $m \ge N \in \mathbb{N}$, we would have

$$(4.6) \quad 0 = \| A_m x' - A_m x'' \| \ge \alpha_m (\| x' - x'' \|) \Longrightarrow \| x' - x'' \| = 0 .$$

which is in contradiction with $x' \neq x''$. Q.E.D.

The assumption (2.1) in Theorem 2.1 can be verified by help of the following

<u>Lemma 4.3</u>. Let the Banach space X have the property $(\pi)_C$ and let the operator $A : X \longrightarrow X$ have the property

$$(4.7)\quad X_i = P_i(X) \Longrightarrow A(X_i) \supset X_i\ , \qquad i = 1, 2, 3, \dots .$$

Then the relation

$$(4.8)\qquad \overline{\bigcup_{n=1}^{\infty} A_n(X)} = X \qquad (\text{where } A_n = P_n A P_n\ )$$

is valid.

<u>Proof</u>. Because $P_n^2 = P_n$, we have, using (4.7) for $\forall n \in \mathbf{N}$

$$(4.9)\quad X \supset A(X_n) = AP_n(X) \supset P_n(X) = X_n \Longrightarrow P_n X \supset A_n(X) =$$
$$= P_n A P_n(X) \supset P_n^2(X) = P_n(X) = X_n$$

and therefore

$$(4.10)\ \bigcup_{n=1}^{\infty} X_n \supset \bigcup_{n=1}^{\infty} A_n(X) \supset \bigcup_{n=1}^{\infty} X_n \Longrightarrow X = \overline{\bigcup_{n=1}^{\infty} X_n} \supset \overline{\bigcup_{n=1}^{\infty} A_n(X)} \supset \overline{\bigcup_{n=1}^{\infty} X_n} = X$$

so that we have

$$\overline{\bigcup_{n=1}^{\infty} A_n(X)} = X$$

with $A_n = P_n A P_n$. Q.E.D.

<u>Remark 4.1</u>. We see by Definition 1.2 that if $A : X \longrightarrow X$ is surjective, Theorem 3.1 gives sufficient conditions for the PW-solvability (PS-solvability resp.) of the equation (1.1) in general normed linear spaces X which need not be reflexive. In [4], some applications are given for the case of reflexive Banach spaces.

R e f e r e n c e s

[1] R. ZEZULA: Über eine Näherungsmethode zur Lösung gewisser Randwertaufgaben der Reaktortheorie, Comment.Math.Univ.Carolinae 3(1962),37-60.

[2] R. ZEZULA: Convergence of algorithms based on projection methods (in Czech).Preprint-Report ÚVT ČVUT 1972.Communicated on the 1st Symposium Algorithms in Computing Technique,Štrbské Pleso,Czechoslovakia,November 1972.

[3] R. ZEZULA: Convergence and error estimation for quasi-projection methods in Banach spaces having the uniform approximation property (in Czech). Submitted to Aplikace matematiky (1973).

[4] W.V. PETRYSHYN: Projection methods in nonlinear numerical functional analysis, Journ.of Mathematics and Mechanics vol.17,no 4(1967),353-372.

[5] R. ZEZULA: On an approximative method for solving some boundary value problems in the nuclear reactor theory, CSc-Dissertation,Prague 1967.Report ÚJV ČSAV 1785/67.

[6] R. ZEZULA: On some optimal regulation problems in reactor physics, Preprint-Report ÚVT ČVUT 1973.

Communicated on the 5th Colloquium on Reactor Physics,Olbersdorf GDR,March 1973.

Matematický ústav
Karlova universita
Sokolovská 83, Praha 8
Československo

(Oblatum 27.11.1973)