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A NOTE ON SUBOBJECTS DEFINED BY LIMIT CONSTRUCTION x)

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Abstract: In this paper we shall show that a definition of subobjects based on a limit construction cannot give anything else than the well-known notion of simultaneous equalizers.

Key words: Simultaneous equalizer, monomorphism having L-property.

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The equalizer and the simultaneous equalizer are morphisms occurring in a limit of diagrams

$$A \rightrightarrows B \quad \text{and} \quad \begin{tikzcd} & B_i \\ A \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & \cdots\cdots\cdots \\ & B_j \end{tikzcd}$$

(More exactly, a simultaneous equalizer of a family of pairs of morphisms $f_i , g_i : A \longrightarrow B_i , i \in I$ is a morphism $u :$ $: X \longrightarrow A$ such that

a) $f_i u = g_i u$ for every $i \in I$,

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x) The results in this paper are a part of my thesis.

b) if $f_i u' = g_i u'$ for every $i \in I$, then there is a unique θ such that $u' = u\theta$.
A morphism is said to be a simultaneous equalizer if it is a simultaneous equalizer of a family of pairs.

An equalizer is clearly a simultaneous equalizer. On the other hand, a simultaneous equalizer in a category with equalizers is an intersection of a family of equalizers. Adding to the observation that in a category with products, equalizers are closed under intersections, we see that in a complete category, simultaneous equalizers coincide with equalizers.)

This suggests an obvious generalization of the limit approach for the notion of equalizers.

First, observe that it would, however, not make sense to define subobjects quite generally as monomorphisms occurring among λ^k's in limits $(\lambda^k : \ell \longrightarrow D(k))_{k \in |K|}$. Really, the pullback diagram

$$\begin{tikzcd} {} \arrow[r, "\alpha"] \arrow[d, "1"'] & {} \arrow[d, "1"] \\ {} \arrow[r, "\alpha"] & {} \end{tikzcd}$$

shows that such is any monomorphism.

Hence, let us confine ourselves to those λ^{k_0} in limits $(\lambda^k : \ell \longrightarrow D(k))_{k \in |K|}$ which necessarily have to be monomorphisms by the nature of the underlying category K of the diagram. Let us start with the following auxiliary

Definition. Let $\mathcal{K}$ be a small category. An object k_0 is said to be at a monomorphism inducing position in $\mathcal{K}$ (abbreviated: " k_0 is MIP in $\mathcal{K}$ ") if, for each limit $(\lambda^k : \ell \longrightarrow D(k))_{k \in |K|}$ of a diagram $D : \mathcal{K} \longrightarrow \mathcal{A}$, λ^{k_0} is a monomorphism.

The notion we are going to study is given in the following

Definition. A monomorphism μ in $\mathcal{A}$ is said to have the L-property if it occurs as λ^{k_0} in a limit

$$(\lambda^k : \ell \longrightarrow D(k))_{k \in |K|}$$

where k_0 is MIP in $\mathcal{K}$ and $\mathcal{K}$ is a small category.

Theorem. A monomorphism μ in $\mathcal{A}$ has L-property if and only if it is a simultaneous equalizer.

For proving the theorem, at first let us list some formally less general properties (which are closer to the definition of simultaneous equalizer we started with).

Definition. A monomorphism μ in $\mathcal{A}$ is said to have the Li -property $(i = 1, 2, 3)$ if it occurs as λ^{k_0} in a limit

$$(\lambda^k : \ell \longrightarrow D(k))_{k \in |K|}$$

where

(L1) for every $k \in |\mathcal{K}|$, $\mathcal{K}(k_0, k) \neq \emptyset$

(L2) for every $k \in |\mathcal{K}|$, $\mathcal{K}(k_0, k) \neq \emptyset$

$\mathcal{K}(k_0, k_0) = \{id_{k_0}\}$ and for $k \neq k_0$, $\mathcal{K}(k, k_0) = \emptyset$

(L3) $\mathcal{K}(k, \ell) \neq \emptyset$ if and only if $k = k_0$,

$\mathcal{K}(\mathscr{k}_0, \mathscr{k}_0) = \{id_{\mathscr{k}_0}\}$.

The theorem will be gradually proved by the following implications on properties indicated in brackets:

(L) $\Longrightarrow$ (L1) $\Longrightarrow$ (L2) $\Longrightarrow$ (L3) $\Longrightarrow$ (Simultaneous equalizer).

The first implication is a direct consequence of the following

Lemma. Let $\mathscr{k}_0, \mathscr{k} \in |\mathcal{K}|$ be such that $\mathcal{K}(\mathscr{k}_0, \mathscr{k}) = \emptyset$. Then $\mathscr{k}_0$ is not MIP in $\mathcal{K}$.

Proof. Let $\mathscr{k}_0, \mathscr{k} \in |\mathcal{K}|$ be such that $\mathcal{K}(\mathscr{k}_0, \mathscr{k}) = \emptyset$. Take a category $\mathcal{A}$ with products and such that there are $\alpha, \beta : a \longrightarrow b$, $\alpha \neq \beta$.

Let s be the singleton(the product of the void system) of $\mathcal{A}$. Let

$$A = \{\mathscr{k} \in |\mathcal{K}| \mid \mathcal{K}(\mathscr{k}_0, \mathscr{k}) \neq \emptyset\}$$

and

$$B = |\mathcal{K}| \setminus A \ .$$

$B \neq \emptyset$ by the assumption. Define a relation R between the members of B :

$$\mathscr{k} R \mathscr{k}' \Longleftrightarrow \exists\, \mathscr{k} \longrightarrow \mathscr{k}' \quad \text{in } \mathcal{K} \ .$$

Let $\sim$ be the equivalence generated by R . Let $M = B/\sim$.

Define a diagram $D : \mathcal{K} \longrightarrow \mathcal{A}$ as follows

$$D(\mathscr{k}) = \begin{cases} s & \text{for } \mathscr{k} \in A \\ b & \text{for } \mathscr{k} \in B \end{cases} ,$$

and for $\mu : \mathscr{k} \longrightarrow \mathscr{k}'$

$$D(\mu) = \begin{cases} id_{\mathfrak{s}} & \text{for } \mathfrak{k}, \mathfrak{k}' \in A \\ id_{\mathfrak{b}} & \text{for } \mathfrak{k}, \mathfrak{k}' \in B \\ \sigma & \text{for } \mathfrak{k} \in B, \mathfrak{k}' \in A . \end{cases}$$

(Note that, by the definition of A and B there does not occur the case where $\mathfrak{k} \in A$, $\mathfrak{k}' \in B$ and σ in the third case is the unique morphism of $\mathfrak{b}$ to $\mathfrak{s}$.)

Let $\mathfrak{b}^M$ be a product with projections $\pi_m : \mathfrak{b}^M \longrightarrow \mathfrak{b}$. For $\mathfrak{k} \in B$ we will denote $[\mathfrak{k}]$ the equivalence class in M represented by $\mathfrak{k}$.

Define a family $(\lambda^{\mathfrak{k}} : \mathfrak{b}^M \longrightarrow D(\mathfrak{k}))_{\mathfrak{k} \in |K|}$ as follows:

for $\mathfrak{k} \in A$, $\lambda^{\mathfrak{k}} : \mathfrak{b}^M \longrightarrow D(\mathfrak{k}) = \mathfrak{s}$ is just the unique morphism to $\mathfrak{s}$, designated by $\wp$,

and for $\mathfrak{k} \in B$, $\lambda^{\mathfrak{k}} = \pi_{[\mathfrak{k}]} : \mathfrak{b}^M \longrightarrow D(\mathfrak{k}) = \mathfrak{b}$.

We will prove that the family $(\lambda^{\mathfrak{k}})$ is just a limit for the diagram D .

At first, $(\lambda^{\mathfrak{k}})$ is a compatible family. Really, let $\mu : \mathfrak{k} \longrightarrow \mathfrak{k}'$ (consequently $\mathfrak{k} \sim \mathfrak{k}'$ if $\mathfrak{k}, \mathfrak{k}' \in B$), we have

if $\mathfrak{k}, \mathfrak{k}' \in A$, $\quad D(\mu)\lambda^{\mathfrak{k}} = id_{\mathfrak{s}} \cdot \wp = \wp = \lambda^{\mathfrak{k}'}$

if $\mathfrak{k}, \mathfrak{k}' \in B$, $\quad D(\mu)\lambda^{\mathfrak{k}} = id_{\mathfrak{b}} \cdot \pi_{[\mathfrak{k}]} = \pi_{[\mathfrak{k}']} = \lambda^{\mathfrak{k}'}$

if $\mathfrak{k} \in B$, $\mathfrak{k}' \in A$, $D(\mu)\lambda^{\mathfrak{k}} = \sigma\pi_{[\mathfrak{k}]} = \wp = \lambda^{\mathfrak{k}'}$

(because $\wp$ is the unique morphism $: \mathfrak{b}^M \longrightarrow \mathfrak{s}$).

Now let $(\tau^{\mathfrak{k}} : X \longrightarrow D(\mathfrak{k}))_{\mathfrak{k} \in |K|}$ be a compatible family. Observe that if $k, k' \in B$, $\mu : k \longrightarrow k'$, then $\tau^{k'} = D(\mu) . \tau^{k} = id_{b} \tau^{k} = \tau^{k}$. Consequently, if $k \sim k'$, then $\tau^{k} = \tau^{k'}$, and we denote the common value by $\varphi_{[k]}$. From which we see that the family $(\tau^{k})_{k \in |K|}$, more exactly, $(\tau^{k})_{k \in B}$ induces a family $(\tau^{m})_{m \in M}$. By the definition of the product b^{M}, there is a unique morphism $\gamma : x \longrightarrow b^{M}$ such that $\pi_m \gamma = \varphi_m$.

γ is just the required morphism in the definition of limit. Really, observing that for $k \in A$, $\tau^{k} : x \longrightarrow D(k) = b$ is just the unique morphism from x to b, designated by φ', we have

$$\text{for } k \in A, \ \lambda^{k}\gamma = \varphi\gamma = \varphi' = \tau^{k} ,$$

$$\text{and for } k \in B, \ \lambda^{k}\gamma = \pi_{[k]}\gamma = \varphi_{[k]} = \tau^{k} .$$

The uniqueness of γ follows by the uniqueness in the definition of the product b^{M}.

At last it suffices to show that φ is not a monomorphism. Let $\Delta : b \longrightarrow b^{M}$ be the diagonal map. Recollect that Δ is a coretraction and $\alpha \neq \beta$, so $\Delta\alpha \neq \Delta\beta$, while $\varphi(\Delta\alpha) = \varphi(\Delta\beta)$ because there is only a unique morphism from a to b. It finishes the proof of the lemma.

The last implication is easily proved as follows:

Let μ be a monomorphism having L3-property. If $(D(\eta))_{\eta \in \langle k_0, k \rangle}$ for every $k \in \mathcal{K}$ has only two members, there is nothing to prove. To simplify the notation in the further proof, let us assume that there is only one $k_1 \in |\mathcal{K}|$ such that $\langle k_0, k_1 \rangle$ has two members. The proof of the general case follows the same line.

Take $\eta_1 \in \langle k_0, k_1 \rangle$, then it is clear that μ is a simultaneous equalizer of the diagram

$$\begin{tikzcd}
 & D(k_1) \\
D(k_0) \arrow[ur, bend left] \arrow[ur] \arrow[dr] \arrow[dr, bend right] & \\
 & D(k)
\end{tikzcd}$$

where $k \neq k_1$ and every $\eta \in \langle k_0, k_1 \rangle$, $\eta \neq \eta_1$ corresponds a pair $(D(\eta_1), D(\eta))$.

Now we prove the implications in the middle.

(L1) $\Longrightarrow$ (L2)

Let μ be a monomorphism occurring as λ^{k_0} in a limit $(\lambda^{k} : \ell \longrightarrow D(k))_{k \in |K|}$, where for $k \in |K|$, $K(k_0, k) \neq \emptyset$.

Construct a category $\widetilde{K}$ as follows:

$$|\widetilde{K}| = |K| \cup \{m\}, \qquad m \notin |K|$$

for

$$k, k' \in |K|, \quad \widetilde{K}(k, k') = K(k, k')$$

$$k \neq m, \quad \widetilde{K}(m, k) = \{(m, \alpha) \mid \alpha : k_0 \longrightarrow k\}$$

$$\widetilde{K}(k, m) = \emptyset$$

and $\quad \widetilde{K}(m, m) = \{id_m\}$.

The new morphisms are composed by the formula

$$\beta \cdot (m, \alpha) = (m, \beta \cdot \alpha).$$

Clearly $\widetilde{K}$ has the property in (L2).

For the diagram $D : K \longrightarrow \mathcal{A}$ we construct a diagram

$$\tilde{D} : \tilde{K} \longrightarrow \mathcal{A}$$

as follows

$$\tilde{D}|_K = D\,, \quad \tilde{D}(n) = D(k_0)$$

and $$\tilde{D}(n, \alpha) = D(\alpha)\ .$$

Now it suffices to show that the family

$$(\tilde{\lambda}^k : \ell \longrightarrow D(k))_{k \in |\tilde{K}|}$$

defined by

$$\tilde{\lambda}^k = \lambda^k \quad \text{for} \quad k \neq n$$
$$\tilde{\lambda}^n = \lambda^{k_0}$$

is a limit of $\tilde{D}$.

At first $(\tilde{\lambda}^k)_{k \in |\tilde{K}|}$ is compatible because:

for $\eta : k \longrightarrow k'$ where $k, k' \neq n$ we have

$$\tilde{D}(\eta)\tilde{\lambda}^k = D(\eta)\lambda^k = \lambda^{k'} = \tilde{\lambda}^{k'}$$

and

$$D((n, \alpha))\tilde{\lambda}^n = D(\alpha)\lambda^{k_0} = \lambda^k = \tilde{\lambda}^k\ .$$

The second condition of limit is clear.

(L2) $\Longrightarrow$ (L3).

Let μ be a monomorphism occurring as λ^{k_0} in a limit $(\lambda^k : \ell \longrightarrow D(k))_{k \in |K|}$ where

$$K(k_0, k_0) = \{id_{k_0}\}$$

$$\mathcal{K}(k_0, k) \neq \emptyset \quad \text{for every} \quad k \in |\mathcal{K}|$$

and $\mathcal{K}(k, k_0) = \emptyset$ for every $k \neq k_0$.

Construct a category $\mathcal{K}'$ as follows:

$$|\mathcal{K}'| = |\mathcal{K}|$$

$$\mathcal{K}'(k_0, k) = \mathcal{K}(k_0, k)$$

$$\mathcal{K}'(k', k) = \emptyset \text{ for every } k' \neq k_0$$

and the composition is defined as in $\mathcal{K}$.

Clearly $\mathcal{K}'$ has the property in (L3).

For the diagram $\mathcal{D} : \mathcal{K} \longrightarrow \mathcal{A}$ we construct a diagram $\mathcal{D}' : \mathcal{K}' \longrightarrow \mathcal{A}$ defined by $\mathcal{D}' = \mathcal{D}|_{\mathcal{K}'}$.

Now we show that the family $(\lambda^{k} : \ell \longrightarrow \mathcal{D}(k) = \mathcal{D}'(k))_{k \in |K|}$ is a limit of $\mathcal{D}$ if and only if it is a limit of $\mathcal{D}'$.

It suffices to prove only one implication.

Let $(\lambda^{k} : \ell \longrightarrow \mathcal{D}(k))_{k \in |K|}$ be a limit f $\mathcal{D}$ as we started. Then (λ^{k}) is clearly compatible relative to $\mathcal{D}'$. Now let (τ^{k}) be compatible relative to $\mathcal{D}$, then, for $\eta : k \longrightarrow k'$ in $\mathcal{K}'$,

$$\mathcal{D}(\eta)\tau^{k} = \mathcal{D}(\eta)\mathcal{D}'(\alpha)\tau^{k_0} = \mathcal{D}(\eta)(\alpha)\tau^{k_0} = \mathcal{D}(\eta)\mathcal{D}(\alpha)\tau^{k_0} = \tau^{k'}.$$

Now, the statement readily follows.

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R e f e r e n c e s

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