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ON CHANGES OF INPUT/OUTPUT CODING II

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Abstract: This paper is an immediate continuation of the paper "On changes of input-output coding I" also published in this journal.

The structure given by a formalization of the intuitive notion of changing output coding is studied. It turns out that this formalization yields a correspondence between the Blum's complexity measures and the weak complexity measures.

Key words: enumeration of partial recursive functions, acceptable enumeration, complexity measure.

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§ 4. σ-dependence

In this paragraph we investigate the structure given by σ-dependence. There are some features in which this structure resembles the one given by i-dependence. We prove that

1) there is an σ-class maximal wrt σ-dependence,

2) the property of being an acceptable enumeration is hereditary wrt σ-dependence,

3) σ-classes of acceptable enumerations can be characterized in a way similar to the characterization of i-classes.

There are, however, features in which σ-dependence differs from i-dependence. For example, σ-classes seem to be too wide. That is why we introduce a more restrictive notion of $\varkappa\sigma$-dependence.

The paragraph is completed with an application of the concept of $\varkappa\sigma$-dependence in the abstract complexity theory.

Theorem 4.1. There is an σ-class maximal wrt σ-dependence.

Proof. Let $\langle\ \rangle : N \times N \longrightarrow N$ be a recursive pairing function. We define $\langle x, y, z\rangle = \langle\langle x, y\rangle, z\rangle$. Let φ be an acceptable enumeration and let $P_0, P_1, P_2, \ldots$ be an effective enumeration of algorithms evaluating the functions $\varphi_0, \varphi_1, \varphi_2, \ldots$. (The use of $P_0, P_1, \ldots$ and of the steps of P_i in this proof is a bit informal. The proof can, however, be entirely formalized e.g. with the aid of an abstract complexity measure - cf. Definition 4.2 .)

Observe that for every effective enumeration α , there is a partial recursive σ such that $\sigma(\langle i, x\rangle) \simeq \alpha_i(x)$ for all $i, x \in N$. We can therefore define a r.e. set $A \subseteq N$ as follows:

$\langle\langle i, x\rangle, \langle j, t, y\rangle\rangle \in A$ iff $P_j(\langle i, x\rangle)$ stops in the t-th step and $P_j(\langle i, x\rangle) = y$.

Let W_i denote $D\varphi_i$ for all $i \in N$. Since A is a r.e. set and since φ is acceptable, a recursive g exists such that

$$\langle x, y\rangle \in A \Longleftrightarrow y \in W_{g(x)} \quad \text{for all } x, y \in N .$$

Obviously

$\langle j, t, y \rangle \in W_{g(\langle i,x \rangle)}$ iff $P_j(\langle i, x \rangle)$ stops in the t-th step and $P_j(\langle i, x \rangle) = y$.

We define $\psi_i(x) \simeq g(\langle i, x \rangle)$ for all $i, x \in N$ and conclude the proof by showing that $[\psi]^\sigma$ is a maximal σ-class, i.e. we show that for every effective enumeration α there is an $h \in \mathcal{O}$ such that $\alpha \leq^\sigma \psi$ via h .

Choose an arbitrary effective enumeration α . There is a $j_0 \in N$ such that $(\forall i, x)[\varphi_{j_0}(\langle i, x \rangle) \simeq \alpha_i(x)]$.

Every r.e. set B can be interpreted as a relation. Integers x, y are in the relation iff $\langle x, y \rangle \in B$. The relation is called single-valued iff $\langle x, y_1 \rangle \in B$ & $\langle x, y_2 \rangle \in B \Longrightarrow y_1 = y_2$ for every $x, y_1, y_2 \in B$. The single-valuedness theorem (cf. [1] § 5.7) asserts that there is a recursive $\varkappa$ such that $W_{\varkappa(x)}$ is single-valued for all $x \in N$ and for W_x single-valued is $W_{\varkappa(x)}$ equal to W_x . We use this function $\varkappa$. Then we can define

$$h(x) = y \quad \text{iff} \quad (\exists t)(\langle j_0, t, y \rangle \in W_{\varkappa(x)})$$

h is a (partial) function by single-valuedness of $W_{\varkappa(x)}$ and is partial recursive by the Projection theorem (cf. [1] § 5.4). Furthermore $Rh = N$. (E.g. singletons of the form $\{\langle j_0, 0, y \rangle\}$ ensure this fact.)

Hence h is an σ-convention and it remains to prove that

$$\alpha_i = h\psi_i \quad \text{for all} \quad i \in N .$$

But for every $i, x, y \in N$

$\alpha_i(x) = y \Longleftrightarrow (\exists t)[P_{j_0}(\langle i,x\rangle)$ stops in the t-th step and $P_{j_0}(\langle i,x\rangle) = y] \Longleftrightarrow$

$\Longleftrightarrow (\exists t)[\langle j_0,t,y\rangle \in W_{g(\langle i,x\rangle)}] \Longleftrightarrow (\exists t)[\langle j_0,t,y\rangle \in W_{hg(\langle i,x\rangle)}]$

$\Longleftrightarrow hg(\langle i,x\rangle) = y \Longleftrightarrow h\psi_i(x) = y$.

The theorem follows.

<u>Theorem 4.2.</u> Let φ, ψ be two enumerations, $\varphi \geq^{\sigma} \psi$ and let φ be acceptable. Then ψ is acceptable.

<u>Proof</u>. Analogous to the proof of Theorem 3.14 - part 2.

<u>Note 4.3</u>. Every enumeration α can be treated as a function of two variables $\lambda ix[\alpha_i(x)]$. So $R\alpha$ will denote the range of the function $\lambda ix[\alpha_i(x)]$.

<u>Theorem 4.4</u>. Let φ, ψ be two acceptable enumerations. Then

$[\varphi \equiv^{\sigma} \psi] \Longleftrightarrow [$ there is a recursive permutation p such that $p\varphi_i = \psi_i$ for all $i \in N]$.

<u>Proof</u>. $\Longleftarrow$: Immediate.

$\Longrightarrow$: Let $\varphi \leq^{\sigma} \psi$ via h and $\psi \leq^{\sigma} \varphi$ via f . Then $\varphi_i = hf\varphi_i$ and $\psi_i = fh\psi_i$ for all $i \in N$. $fh = hf = id$, since $R\varphi = R\psi = N$. Consequently f is a recursive permutation.

σ -dependence of acceptable enumerations can be characterized in a simple manner as the following theorem shows.

Theorem 4.5. Let φ, ψ be acceptable enumerations. Then

$[\varphi \geq^{\sigma} \psi] \Longleftrightarrow [\varphi_i(x) \simeq \varphi_j(y) \Longrightarrow \psi_i(x) \simeq \psi_j(y)$ for all $i, j, x, y \in N]$.

Proof. 1) $\Longrightarrow$: Immediate.

2) $\Longleftarrow$: There is an $i_0 \in N$ such that $\varphi_{i_0} = id$. Let us define $h = \psi_{i_0}$. We prove that $\psi_i = h\varphi_i$ for all $i \in N$.

a) Let $\varphi_i(x)\downarrow$. Then $\varphi_i(x) = y$ for some $y \in N$. Apparently $\varphi_i(x) = \varphi_{i_0}(y)$ and hence $\psi_i(x) \simeq \psi_{i_0}(y) \simeq h(y) \simeq$ $\simeq h(\varphi_i(x))$.

b) Let $\varphi_i(x)\uparrow$. We show that $\psi_i(x)\uparrow$. Assume on the contrary that $\psi_i(x) = y_0$ for some y_0 . Then $\psi_k(x) = y_0$ whenever $\varphi_k(x)\uparrow$. $Rh \cup \{y_0\} = N$, as ψ is acceptable and $R\psi = N$. Apparently there exists an $m \in N$ such that $h(m) \neq y_0$.

There is a $j_0 \in N$ such that

$$\varphi_j(x) \simeq \begin{cases} m & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Then by assumption

$$\psi_{j_0}(x) = \begin{cases} h(m) & \text{if } \varphi_x(x)\downarrow \\ y_0 & \text{if } \varphi_x(x)\uparrow \end{cases}$$

This is a contradiction. Consequently $\psi_i(x)\uparrow$ and $\psi_i = \mathfrak{h}\varphi_i$ for all $i \in N$. It implies $R\mathfrak{h} = N$ and the theorem follows.

Corollary 4.6. Let φ, ψ be acceptable enumerations. Then

[there is a recursive permutation p such that $p\varphi_i = \psi_i$ for all $i \in N$] $\Longleftrightarrow$

$\Longleftrightarrow$ $[\varphi_i(x) \simeq \varphi_j(y) \Longleftrightarrow \psi_i(x) \simeq \psi_j(y)$ for all $i, j, x, y \in N]$.

As it can be expected, the structures given on the class of enumerations by i-dependence and σ-dependence respectively differ in many eassential properties. For example, in contrast to Corollary 3.15 there are σ-conventions f, g such that for every acceptable enumeration φ, and enumerations α, β, σ-dependent on φ via f and g respectively, $[\alpha]^{\sigma}, [\beta]^{\sigma}$ have no least upper bound.

In spite of the essential difference between changes of input and output codings, it seems to be natural to choose the formalization of changes of output coding so that two enumerations would be equivalent iff they equal up to a recursive permutation of outputs, i.e. similarly as the concept of i-dependence was chosen (cf. Theorem 3.8). In this sense, however, the concept of σ-dependence proves to be too weak.

Fact 4.7. There are two σ-equivalent enumerations

α, β such that $(\forall i \in N)(p\alpha_i = \beta_i)$ does not hold for any recursive permutation p.

<u>Proof</u>. Recall the recursion theoretic notion of recursive isomorphism. Two sets $A, B \subseteq N$ are recursively isomorphic iff there is a recursive permutation p such that $p(A) = B$.

Observe that if $\alpha \leq^{\sigma} \beta$ via some recursive permutation p, then $R\alpha$ is recursively isomorphic to $R\beta$. Thereby, to prove the fact it suffices to exhibit σ-equivalent α, β such that $R\alpha$ and $R\beta$ are not recursively isomorphic.

Let α be an enumeration such that $R\alpha$ is an r.e. nonrecursive set and there is an infinite recursive set C in the complement of $R\alpha$ (i.e. $R\alpha$ is not simple set).

Let β be an enumeration such that $R\beta$ is an infinite recursive set with infinite complement. Then $R\alpha$ and $R\beta$ are not recursively isomorphic (cf. [1]). We show that $\alpha \equiv^{\sigma} \beta$.

$R\alpha$ and $R\beta$ are infinite r.e. sets. Hence a partial recursive 1-1 function ψ exists such that $D\psi = R\alpha$ and $\psi(R\alpha) = R\beta$. Similarly there are partial recursive σ, φ such that $D\sigma = C \;\&\; R\sigma = N$ and $D\varphi = \overline{R\beta} \;\&$ $\& \; R\varphi = N$.

Then the functions

$$h(x) \simeq \begin{cases} \sigma(x) & \text{if } x \in C \\ \psi(x) & \text{otherwise} \end{cases}$$

and

$$f(x) \simeq \begin{cases} \varphi(x) & \text{if } x \in \overline{R\beta} \\ \psi^{-1}(x) & \text{if } x \in R\beta \end{cases}$$

are evidently σ -conditions and $\beta \leq^{\sigma} \alpha$ via h , $\alpha \leq^{\sigma} \beta$ via f .

This concludes the proof.

In practice, when changing the output coding, we implicitly demand the possibility of deciding effectively what outputs will be without interpretation and what outputs will code numbers in the new coding. That is why the following concept of $\varkappa\sigma$ -dependence does not seem to be too restrictive.

The $\varkappa\sigma$ -classes coincide with the classes of enumerations equivalent up to a "permutation of outputs".

Furthermore, the notion of $\varkappa\sigma$ -dependence yields a correspondence between Blum's complexity measures and the weak complexity measures introduced by I.M. Havel and G. Ausiello (cf. [3],[4],[5]).

<u>Definition 4.1</u>. Let α , β be two enumerations. Then we define:

1) α $\varkappa\sigma$ <u>-depends on</u> β <u>via</u> f ($\alpha \leq^{\varkappa\sigma} \beta$ via f) iff $\alpha \leq^{\sigma} \beta$ via f and Df is a recursive set.

2) α $\varkappa\sigma$ <u>-depends on</u> β ($\alpha \leq^{\varkappa\sigma} \beta$) iff there is an $f \in \mathcal{O}$ such that $\alpha \leq^{\varkappa\sigma} \beta$ via f .

3) α <u>is</u> $\varkappa\sigma$ <u>-equivalent to</u> β ($\alpha \equiv^{\varkappa\sigma} \beta$) iff

$\alpha \leq^{ro} \beta$ and $\beta \leq^{ro} \alpha$.

4) $[\alpha]^{ro}$ denotes ro-equivalence class containing α .

Note 4.8. Part 4) of the previous definition makes sense, since $\equiv^{ro}$ is really an equivalence relation as can easily be verified.

Theorem 4.9. Let α, β be two enumerations. Then $[\alpha \equiv^{ro} \beta] \Longleftrightarrow [$ there is a recursive permutation h such that $h\alpha_i = \beta_i$ for all $i \in N$ $]$.

Proof. 1) $\Longleftarrow$: Immediate.

2) $\Longrightarrow$: Let $\alpha \geq^{ro} \beta$ via f, $\beta \geq^{ro} \alpha$ via g . Evidently $gf\alpha_i = \alpha_i$ and $fg\beta_i = \beta_i$ for all $i \in N$. Therefore

$(*)$ $(\forall x \in R\alpha)[gf(x) = x]$ and $(\forall x \in R\beta)[fg(x) = x]$.

We define the sets A, B as follows:

$$A = \{x \in Df : f(x) \in Dg \ \& \ gf(x) = x\}$$
$$B = \{x \in Dg : g(x) \in Df \ \& \ fg(x) = x\} .$$

Apparently A and B are recursive and $A \supset R\alpha$, $B \supset R\beta$ by $(*)$. Moreover, f is 1-1 on A and $f(A) = B$. We prove that

$$(**) \qquad card\,\overline{A} = card\,\overline{B} .$$

$f^{-1}(\overline{B}) \subset \overline{A}$ and $card\,\overline{B} = card[f^{-1}(\overline{B})]$ as $Rf = N$. Hence $card\,\overline{B} = card[f^{-1}(\overline{B})] \leq card\,\overline{A}$. Analogously $card\,\overline{A} \leq card\,\overline{B}$. $(**)$ therefore holds and since A, B are recursive, a partial recursive 1-1 function ψ exists such that $D\psi = \overline{A}$ and $\psi(\overline{A}) = \overline{B}$.

We can define

$$h(x) \simeq \begin{cases} f(x) & \text{if } x \in A \\ \psi(x) & \text{if } x \in \overline{A}\ . \end{cases}$$

h is recursive permutation and $f(x) = h(x)$ for $x \in R\alpha$. Consequently $h\alpha_i = \beta_i$ for all $i \in N$.

The additional condition in the definition of ro-dependence causes that some "nice" properties of i-dependence are lost. E.g. Theorem 4.5 does not hold for ro-dependence. Another example is the following theorem which contrasts with Theorem 4.1.

<u>Theorem 4.10</u>. For every acceptable enumeration φ, there is an acceptable enumeration ψ such that no upper bound (wrt $\leq^{ro}$) of ψ, φ exists.

<u>Proof</u>. There is an $i_0 \in N$ such that

$$\varphi_{i_0}(x) \simeq \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

We define ψ as follows:

$$\psi_i(x) \simeq \begin{cases} 1 & \text{if } i = i_0 \\ \varphi_i(x) & \text{otherwise.} \end{cases}$$

ψ is evidently acceptable. Assume that there is α and appropriate $f, g \in \mathcal{O}$ such that

$$\varphi \leq^{\kappa\sigma} \alpha \quad \text{via } f \quad \text{and} \quad \psi \leq^{\kappa\sigma} \alpha \quad \text{via } g .$$

Obviously $R\alpha_{i_0} \subset Dg$ and $\alpha_{i_0}(x)\downarrow$ for every $x \in N$. It implies

$$\alpha_{i_0}(x) \in Df \Longleftrightarrow \varphi_{i_0}(x)\downarrow .$$

Thereby

$$(*) \qquad \varphi_x(x)\downarrow \Longleftrightarrow \alpha_{i_0}(x) \in Df .$$

Since Df is recursive, $(*)$ would give a recursive procedure for deciding whether $\varphi_x(x)\downarrow$. This is a contradiction.

Since the assumption of existence of an upper bound α proves to be contradictory, the theorem follows.

In [3], M. Blum formulated the following, machine-independent definition of complexity measure.

Definition 4.2. Let φ be an acdeptable enumeration, Φ an enumeration. We say that Φ is complexity measure (CM) for φ iff the following two conditions hold:

1) $(\forall i, x \in N)\ [\varphi_i(x)\downarrow \Longleftrightarrow \Phi_i(x)\downarrow]$.

2) There is an $m \in R_3$ such that

$$m(i,x,y) = \begin{cases} 1 & \text{if } \Phi_i(x) = y \\ 0 & \text{otherwise.} \end{cases}$$

The conditions 1),2) are so weak that they are satisfied by all concrete complexity measures. In spite of that, the first condition is a bit restrictive, as there exist nonterminating computations using only finite amount of a resource. (E.g. Turing-machine computations cycling on a finite amount of tape.) The next definition ([4]) reflects the fact.

For purposes of the definition we introduce the following notation.

Notation 4.3. For two arbitrary enumerations φ, Φ,

$\varphi_k(x)\downarrow\downarrow(\Phi)$ denotes $\varphi_k(x)\downarrow$ & $\Phi_k(x)\downarrow$

$\varphi_k(x)\uparrow\downarrow(\Phi)$ denotes $\varphi_k(x)\uparrow$ & $\Phi_k(x)\downarrow$

$\varphi_k(x)\uparrow\uparrow(\Phi)$ denotes $\varphi_k(x)\uparrow$ & $\Phi_k(x)\uparrow$.

Definition 4.4. Let φ be an acceptable enumeration, Φ an enumeration. We say that Φ is a weak complexity measure (WCM) for φ iff the following conditions hold.

1a) $(\forall i, x \in N)[\varphi_i(x)\downarrow \implies \Phi_i(x)\downarrow]$.

1b) There is a $\vartheta \in P_2$ such that

$\forall i, x \in N$:

$$\Phi_i(x)\downarrow \Longrightarrow \vartheta(i,x) = \begin{cases} 1 & \text{if } \varphi_i(x)\downarrow \\ 0 & \text{otherwise.} \end{cases}$$

2) There is an $m \in R_3$ such that

$$m(i,x,y) = \begin{cases} 1 & \text{if } \Phi_i(x) = y \\ 0 & \text{otherwise.} \end{cases}$$

3a) There is a $q \in R_2$ such that for all $i, j, x \in N$

(i) $\varphi_{q(i,j)} = \varphi_j \varphi_i$

(ii) $\varphi_{q(i,j)} \uparrow\downarrow (\Phi)$ if $\varphi_i(x) = y$ & $\varphi_j(y) \uparrow\downarrow (\Phi)$ for some y .

3b) There is an $\varkappa \in R_2$ such that for all $i, j, x \in N$

(i)

$$\varphi_{\varkappa(i,j)}(x) \simeq \begin{cases} \varphi_i(x) & \text{if } x > 0 \\ \varphi_j(x) & \text{if } x = 0 \end{cases}$$

(ii) $\varphi_{\varkappa(i,j)}(x) \uparrow\downarrow (\Phi)$ if either $x > 0$ & $\varphi_i(x) \uparrow\downarrow (\Phi)$ or $x = 0$ & $\varphi_j(x) \uparrow\downarrow (\Phi)$.

The following theorem is due to I.M. Havel.

<u>Theorem 4.11</u>. Φ is WCM for an acceptable enumeration φ iff one of the following conditions holds.

1) Φ is CM for φ .

2) The conditions 1a), 1b), 2) of Definition 4.4 are satisfied by φ and Φ and there is a $p \in R_1$ such that

$\forall i, x \in N$:

$$\varphi_{p(i)}(x) \simeq \begin{cases} \varphi_i(x) \dot{-} 1 & \text{if } \varphi_i(x) > 0 \\ \uparrow\downarrow(\Phi) & \text{if } \varphi_i(x) = 0 \\ \uparrow\uparrow(\Phi) & \text{otherwise.} \end{cases}$$

By this theorem, WCM are more general than CM. $\varkappa\sigma$-dependence gives, however, a tight relation between the two concepts. We prove it in the rest of the paragraph.

<u>Theorem 4.12</u>. Let φ be an acceptable enumeration, Φ a complexity measure for φ. Let ψ be an enumeration $\varkappa\sigma$-dependent on φ (via some σ-convention f).

Then Φ is WCM for ψ.

<u>Proof</u>. ψ is acceptable by Theorem 4.2. We define

$$\vartheta(i,x) \simeq \begin{cases} 1 & \text{if } \varphi_i(x)\downarrow \,\&\, \varphi_i(x) \in Df \\ 0 & \text{if } \varphi_i(x)\downarrow \,\&\, \varphi_i(x) \notin Df \\ \uparrow & \text{otherwise.} \end{cases}$$

Evidently $\vartheta \in P_2$ and ϑ satisfies condition 1b) of Definition 4.4 . 1a) is satisfied trivially, condition 2) holds by the definition of CM.

Furthermore, there are functions $q, \varkappa \in R_2$ such that $\varphi_{q(i,j)} = \varphi_j f \varphi_i$ and

$$\varphi_{\varkappa(i,j)} \simeq \begin{cases} \varphi_i(x) & \text{if } x > 0 \\ \varphi_j(x) & \text{if } x = 0 \end{cases} .$$

The existence of the functions ϱ, $\varkappa$ follows from Church's thesis and the definition of acceptable enumeration. For more formal proof see e.g. [1] § 1.8 .

Since $\psi_{\varrho(i,j)} = f\varphi_{\varrho(i,j)} = f\varphi_j f\varphi_i$ and

$$\psi_{\varkappa(i,j)}(x) \simeq f\varphi_{\varkappa(i,j)}(x) \simeq \begin{cases} f\varphi_i(x) & \text{if } x > 0 \\ f\varphi_j(x) & \text{if } x = 0 , \end{cases}$$

the functions ϱ, $\varkappa$ satisfy conditions 3a), 3b).

Theorem 4.13. Let φ be an acceptable enumeration and Φ a WCM for φ . Then there is an acceptable ψ such that $\psi \geq^{n\sigma} \varphi$ and Φ is CM for ψ .

Proof. We use Theorem 4.11.

1) If Φ is CM for φ , then take $\psi = \varphi$.

2) If Φ is not CM for φ , then the condition 2) of Theorem 4.11 holds. Therefore the functions ϑ, η of the described properties exist. Define:

$$\psi_i(x) \simeq \begin{cases} \varphi_i(x) + 1 & \text{if } \varphi_i(x) \downarrow\downarrow (\Phi) \\ 0 & \text{if } \varphi_i(x) \uparrow\downarrow (\Phi) \\ \uparrow & \text{otherwise.} \end{cases}$$

ψ is effective enumeration. By the definition of acceptable enumeration, a recursive q exists such that $\psi_i = \varphi_{q(i)}$ for all $i \in N$.

We show that $\psi_{p(i)} = \varphi_i$ for all $i \in N$.

a) Let $\varphi_i(x) > 0$. Then

$$\varphi_i(x) = \varphi_{p(i)}(x) + 1 = \psi_{p(i)}(x) .$$

b) Let $\varphi_i(x) = 0$. Then

$\varphi_{p(i)}(x)\uparrow\downarrow(\Phi)$ and consequently $\psi_{p(i)}(x) = 0$.

c) Let $\varphi_i(x)\uparrow$. Then $\varphi_{p(i)}(x)\uparrow\uparrow(\Phi)$ and therefore $\psi_{p(i)}(x)\uparrow$.

So ψ is acceptable by Theorem 3.2.

Apparently ψ and Φ satisfy condition 1) of Definition 4.2 . Moreover, Φ satisfies the condition 2) by assumption. It remains to prove that $\psi \geq^{no} \varphi$.

Let us define

$$h(x) \simeq \begin{cases} x \dot{-} 1 & \text{if } x > 0 \\ \uparrow & \text{if } x = 0 . \end{cases}$$

$Rh = N$ and $Dh = N \setminus \{0\}$ is recursive. Obviously $\psi \geq^{no} \varphi$ via h . The theorem follows.

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