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STRONG EMBEDDINGS INTO CATEGORIES OF ALGEBRAS OVER A MONAD, I.

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Abstract: Hedrlín, Isbell, Kučera, Pultr, Trnková and others have intensively investigated full and strong embeddings of concrete categories into categories of algebras. This paper considers the possibility of replacing usual categories of algebras by equational and varietal categories in the sense of Linton. All considerations are carried out for an arbitrary category in the place of the category of sets.

Key words: Equational category, varietal category, U-algebra, monad, algebra over a monad, full embedding, strong embedding, Kan extension, Beck's theorem, absolute limit, split coequalizer.

AMS: 18B15, 18C99 Ref. Ž. 2.725.11,2.725.3.

Full embeddings of concrete categories into categories of algebras were investigated in many papers (e.g. [5],[6], [7] or [18]). In these papers, categories of algebras are categories $\mathfrak{A}(\Delta)$ of all algebras of the type Δ and their homomorphisms, where $\Delta = (\alpha_\beta)_{\beta < \gamma}$ is a set of ordinals indexed by ordinals. Thus categories, such as the category of complete semilattices, complete Boolean algebras and complete homomorphisms or the category of compact Hausdorff spaces and continuous mappings, which are defined by operations, are not categories of algebras in this sense because they are without a rank (the supremum of arities of operations used). Kučera and Hedrlín proved in 1969 (see [9]) that any concrete category can be fully embedded into some $\mathfrak{Alg}(\Delta)$ under the assumption

(M) There is a cardinal m such that every ultrafilter closed under intersections of m elements is trivial.

Under $non(M)$, the category of compact Hausdorff spaces and the category of complete Boolean algebras cannot be fully embedded into any $\mathfrak{Alg}(\Delta)$ (see [10]). It seems that the appearance of the axiom (M) is caused by the fact that the categories $\mathfrak{Alg}(\Delta)$ have a rank. The result of V. Trnková quoted in [9] implies that any concrete category can be fully embedded into a category of "algebras" without a rank. Thus it is reasonable to consider full embeddings into so general categories of algebras to include categories of algebras without a rank. It is natural to take algebras over a monad or algebras in the sense of Linton ([12]). The investigation can be carried out for algebras over arbitrary categories and not only for algebras over the category $\mathbf{Ens}$ of sets.

We shall need the following generalization of the notion of a concrete category. A pair $(\mathbf{M}, \mathbf{U})$ consisting of a category $\mathbf{M}$ and a faithful functor $\mathbf{U}: \mathbf{M} \longrightarrow \mathbf{A}$ is called a category structured over the category $\mathbf{A}$ (see [2]). Since the full embeddability into categories of algebras is not a suitable criterium of "algebraicity", a special class of full embeddings, so called strong embeddings, was defined and dealt with in [18] and [19]. We extend this definition to a general base category $\mathbf{A}$. Further, we introduce nice embeddings which turn out to share many properties of strong embeddings with respect to the "algebraicity". Again, they were actually defined in [19] in a special case.

Let $(\mathcal{M}, U)$ be a category structured over a category $\mathbf{A}$, $(\mathcal{N}, W)$ over $\mathbf{B}$ and $F: \mathbf{A} \longrightarrow \mathbf{B}$ a functor. A full embedding $H: \mathcal{M} \longrightarrow \mathcal{N}$ is called an F-strong embedding if the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{H} & \mathcal{N} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle W} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{B} \end{array}$$

commutes. If H is an F-strong embedding for some $F: \mathbf{A} \longrightarrow \mathbf{B}$, then it is called a strong embedding (see [18] for $\mathbf{A} = \mathbf{B} = \mathbf{Ens}$). An $Id_{\mathbf{A}}$-strong embedding is called a realization (see [16] for $\mathbf{A} = \mathbf{Ens}$ or [2] under the name of a structural functor). An embedding $H: \mathcal{M} \longrightarrow \mathcal{N}$ for which $WH = FU$ is called an F-nice embedding if H is as full as F; i.e. if $f: Hm \longrightarrow Hm'$ is an arrow in $\mathcal{N}$ such that $Wf = Ff_1$ for some $f_1: Um \longrightarrow Um'$, then $f = Hf'$ for some $f': m \longrightarrow m'$ in $\mathcal{M}$. A nice embedding is that which is F-nice for some F.

We can add that pairs F, H of functors $F: \mathbf{A} \longrightarrow \mathbf{B}$, $H: \mathcal{M} \longrightarrow \mathcal{N}$ such that $FU = WH$ are arrows of the category of structured categories considered as a full subcategory of the category of arrows of the category of categories.

In § 1 we recall the notions of a Kan extension, an algebra over a monad and an algebra in the sense of Linton. In § 2, F-strong and F-nice embedding are considered. For a category $(\mathbf{M}, \mathbf{U})$ structured over $\mathbf{A}$ and $F: \mathbf{A} \longrightarrow \mathbf{B}$ there is constructed a canonical embedding into a category of algebras over $\mathbf{B}$ which turns out to be F -strong (F -nice) whenever an F -strong (F -nice) embedding into a category of algebras over $\mathbf{B}$ exists. Nice embeddability of $(\mathbf{M}, \mathbf{U})$ into a category of algebras makes $\mathbf{U}$ to reflect some limits and colimits and if $\mathbf{U}$ has an adjoint, then such a reflection is sufficient for this embeddability as it is shown in § 3. In § 4, we consider in more detail the case $\mathbf{A} = \mathbf{Ens}$ and we touch full embeddings in general case.

§ 1. Preliminaries.

All necessary concepts from the theory of categories can be found in [14]. We recall some of them. Notation used here is taken from [14]. $\mathbf{A}(a, b)$ for $a, b \in \mathbf{A}$ is the set of all arrows $a \longrightarrow b$ in a category $\mathbf{A}$. A natural transformation α from a functor S to R is denoted by $\alpha: S \xrightarrow{\cdot} R$ and $Nat(S, R)$ is the family of all natural transformations from S to R. By a functor, a covariant one is meant.

Given functors $K: \mathbf{M} \longrightarrow \mathbf{C}$ and $T: \mathbf{M} \longrightarrow \mathbf{A}$, a right Kan extension of T along K is a pair $Ran_K T = R: \mathbf{C} \longrightarrow \mathbf{A}$, $\varepsilon: RK \xrightarrow{\cdot} T$ such that for each pair $S: \mathbf{C} \longrightarrow \mathbf{A}$, $\alpha: SK \xrightarrow{\cdot} T$ there is a unique natural transformation $\sigma: S \xrightarrow{\cdot} R$ such that $\alpha = \varepsilon \cdot \sigma K: SK \xrightarrow{\cdot} T$. The assignment $\sigma \longmapsto \varepsilon \cdot \sigma K$ is a bijection $Nat(S, R) \cong$ $\cong Nat(SK, T)$ natural in S; again, this natural bijection determines R from K and T. Let $(c \downarrow K)$ for $c \in \mathbf{C}$ be the comma category and $Q: (c \downarrow K) \longrightarrow \mathbf{M}$ the projection.

$(c \downarrow K)$ has objects $\langle f, m \rangle$, where $f : c \longrightarrow Km$ is an arrow in C and arrows $h : \langle f, m \rangle \longrightarrow \langle f', m' \rangle$ are those arrows $h : m \longrightarrow m'$ in M for which $f' = K(h)f$. Q is defined by

$$\begin{array}{ccc} \langle f, m \rangle & \longmapsto & m \\ \downarrow h & & \downarrow h \\ \langle f', m' \rangle & \longmapsto & m' \end{array}$$

If the composite $(c \downarrow K) \xrightarrow{Q} M \xrightarrow{T} A$ has for each $c \in C$ a limit in A, then R exists and $Rc = \mathrm{Lim}\,((c \downarrow K) \xrightarrow{Q} M \xrightarrow{T} A)$ for each $c \in C$. It is the most frequent case of the appearance of R and this R is called a pointwise right Kan extension.

Let X be a category. Define a category $X^{\S}$, called the subdivision category of X. The objects of $X^{\S}$ are all symbols $x^{\S}$ and $f^{\S}$ for $x \in X$ and f an arrow in X. The arrows of $X^{\S}$ are the identity arrows for these objects, plus for each arrow $f : x \longrightarrow y$ in X two arrows $x^{\S} \longrightarrow f^{\S} \longleftarrow y^{\S}$. The only meaningful compositions for these arrows in $X^{\S}$ are the compositions with one factor an identity arrow. Let X^{op} be the opposite (dual) category for X, Y another category and $D : X^{op} \times X \longrightarrow Y$ a functor. Then D defines a functor $D^{\S} : X^{\S} \longrightarrow Y$ by the assignments indicated in the following figure for a typical $f : x \longrightarrow y$ in X:

$$\begin{array}{ccccccccc} X^{\S} & \quad & x^{\S} & \text{———} & f^{\S} & \text{———} & y^{\S} \\ \downarrow D^{\S} & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Y & & D(x,x) & \xrightarrow{D(f,x)} & D(y,x) & \xleftarrow{D(y,f)} & D(y,y) \end{array}$$

If the functor $D^{\S}$ admits a limit, then this limit is called an end of $D: X^{op} \times X \longrightarrow Y$ and is denoted by $\int_x D(x,x)$.

Let $K: M \longrightarrow C$, $T: M \longrightarrow A$ and for all $m', m \in M$ and all $c \in C$ the power $Tm^{C(c,Km')}$ exists. Then $\langle m', m \rangle \longmapsto Tm^{C(-,Km')}$ is (the object function of) a functor $M^{op} \times M \longrightarrow A^C$. Further, T has a right Kan extension along K if and only if this functor has an end, and this end is the Kan extension $\mathrm{Ran}_K T = \int_m Tm^{C(-,Km)}$ (Ulmer, see [14], p. 239 ex. 5).

A monad $\mathbf{T} = \langle T, \eta, \mu \rangle$ in a category A consists of a functor $T: A \longrightarrow A$ and two natural transformations $\eta : Id_A \xrightarrow{\cdot} T$, $\mu: T^2 \xrightarrow{\cdot} T$ such that $\mu \cdot \eta T = Id_T$, $\mu \cdot T\eta = Id_T$ and $\mu \cdot T\mu = \mu \cdot \mu T$. An algebra over $\mathbf{T}$ (or briefly a $\mathbf{T}$-algebra) is a pair $\langle a, h \rangle$ consisting of an object $a \in A$ and an arrow $h: Ta \longrightarrow a$ of A such that $h \cdot \eta_a = id_a$ and $hT(h) = h \cdot \mu_a$. A morphism $f: \langle a, h \rangle \longrightarrow \langle a', h' \rangle$ of $\mathbf{T}$-algebras in an arrow $f: a \longrightarrow a'$ of A with $fh = h' \cdot T(f)$.

Let $A^{\mathbf{T}}$ be the category of all $\mathbf{T}$-algebras and their morphisms. Categories isomorphic to some $A^{\mathbf{T}}$ are called monadic. The assignments

$$G^{\mathbf{T}}: \begin{array}{ccc} \langle a, h \rangle & \longmapsto & a \\ \downarrow f & & \downarrow f \\ \langle a', h' \rangle & \longmapsto & a' \end{array} \qquad F^{\mathbf{T}}: \begin{array}{ccc} a & \longmapsto & \langle Ta, \mu_a \rangle \\ \downarrow f & & \downarrow Tf \\ a' & \longmapsto & \langle Ta', \mu_{a'} \rangle \end{array}$$

give the functors $G^T : A^T \longrightarrow A$, $F^T : A \longrightarrow A^T$ and F^T is a left adjoint for G^T . Further, T is the monad defined by this adjunction, i.e. $T = G^T F^T$.

By the dualization we obtain comonads, coalgebras over a comonad and comonadic categories,

Let $K : M \longrightarrow A$ have a right Kan extension R_K , ε along itself; $\varphi : Nat(S, R_K) \cong Nat(SK, K)$. Then $\langle R_K, \eta, \mu \rangle$ is a monad in A , where $\eta = \varphi^{-1}(Id_K)$, $\mu = \varphi^{-1}(\varepsilon \cdot R\varepsilon)$ (see [14], p.246 ex.3 or [12] for the poinwise case). This monad is called the codensity monad of K . If K has a left adjoint $F : A \longrightarrow M$, then the codensity monad exists and is equal to the monad defined by the adjunction. The assignment

$$\begin{array}{ccc} m & \longmapsto & \langle Km, \varepsilon_m \rangle \\ \downarrow f & & \downarrow Kf \\ m' & \longmapsto & \langle Km', \varepsilon_{m'} \rangle \end{array}$$

gives the functor $\overline{K} : M \longrightarrow A^{R_K}$. Namely, $\langle Km, \varepsilon_m \rangle$ is an R_K-algebra for each $m \in M$ for $\varepsilon \cdot \mu K = \varepsilon \cdot \varphi^{-1}(\varepsilon \cdot R_K \varepsilon)K = \varphi \varphi^{-1}(\varepsilon \cdot R_K \varepsilon) = \varepsilon \cdot R_K \varepsilon$ and $\varepsilon \cdot \eta K = \varepsilon \cdot \varphi^{-1}(Id_K) = \varphi \varphi^{-1}(Id_K) = Id_K$. Here the definition of the natural bijection φ by means of ε is used. Further, Kf is a morphism of R_K-algebras for the naturality of $\varepsilon : R_K K \overset{\cdot}{\longrightarrow} K$. Clearly $K = G^{R_K}\overline{K}$.

Besides algebras over a monad we shall need algebras arising from a functor $V : X \longrightarrow A$ (see Linton [12]). Let

$V^n = A(n, V-)$ for $n \in A$ and $V^f : V^n \xrightarrow{\cdot} V^k$ be the natural transformation induced by $f : k \longrightarrow n$. Let $a^n = A(n,a)$ for $a, n \in A$, $a^f = A(f,a) : a^n \longrightarrow a^k$ for $f : k \longrightarrow n$ in A and $g^n = A(n,g) : a^n \longrightarrow b^n$ for $g : a \longrightarrow b$ in A. A V-algebra is then defined to be a system $(a, \mathcal{A})$ consisting of an object $a \in A$ and a family $\mathcal{A} = \{\mathcal{A}_{n,k} / n, k \in A\}$ of functions

$$\mathcal{A}_{n,k} : Nat(V^n, V^k) \longrightarrow Ens(a^n, a^k)$$

satisfying the identities

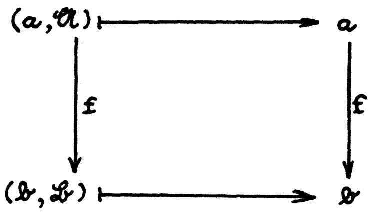
$$\mathcal{A}_{n,k}(V^f) = a^f \quad \text{for } f : k \longrightarrow n$$

$$\mathcal{A}_{n,m}(\theta' \cdot \theta) = \mathcal{A}_{k,m}(\theta')\mathcal{A}_{n,k}(\theta) \text{ for } \theta : V^n \xrightarrow{\cdot} V^k,\ \theta' : V^k \xrightarrow{\cdot} V^m .$$

As V-algebra homomorphisms from $(a, \mathcal{A})$ to $(b, \mathcal{B})$ we admit all arrows $g : a \longrightarrow b$ of A making the diagram

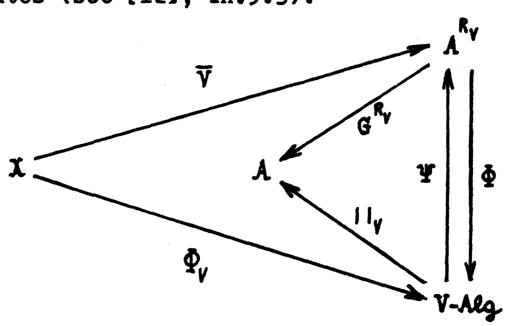
$$\begin{array}{ccc} a^n & \xrightarrow{\quad g^n \quad} & b^n \\ \downarrow{\scriptstyle \mathcal{A}_{n,k}(\theta)} & & \downarrow{\scriptstyle \mathcal{B}_{n,k}(\theta)} \\ a^k & \xrightarrow{\quad g^k \quad} & b^k \end{array}$$

commute for each natural operation θ $Nat(V^n, V^k)$. We write $V\text{-}Alg$ for the resulting category of V-algebras. The assignment $\Phi_V x = (Vx, \mathcal{A}^x)$, where $\mathcal{A}^x_{n,k}(\theta) = \theta_x$ for $\theta : V^n \xrightarrow{\cdot} V^k$, gives a functor $\Phi_V : X \longrightarrow V\text{-}Alg$ ($\Phi_V(f) = Vf$). Further, the assignment



defines the underlying A -object functor $|\ |_V : V\text{-}\mathcal{A}lg \longrightarrow A$. Clearly $V = |\ |_V\, \Phi_V$.

If a functor $V : X \longrightarrow A$ admits a codensity monad R_V, then there exists an isomorphism $\Phi : A^{R_V} \longrightarrow V\text{-}\mathcal{A}lg$ with the inverse $\Psi : V\text{-}\mathcal{A}lg \longrightarrow A^{R_V}$ such that the following diagram commutes (see [12], Th.9.3).



Hence for any monad T in A the category A^T is isomorphic with the category $G^T\text{-}\mathcal{A}lg$ of G^T -algebras. In the case $A = \mathcal{E}ns$ categories $V\text{-}\mathcal{A}lg$ for set valued functors $V :$ $: X \rightarrow \mathcal{E}ns$ are precisely equational categories and categories $\mathcal{E}ns^T$ are varietal categories in the sense of Linton [11]. Varietal categories are equational categories for which the underlying $\mathcal{E}ns$ -object functor has a left adjoint. Categories dual to equational categories were characterized in [3] under the name of quasi-cotripleable categories. The example of an equational category which is not varietal is the category of complete Boolean algebras (see [11]) or the category of complete Boolean algebras with the closure operation (see [8]). If T is a monad in $\mathcal{E}ns$ and we want to determine the operations of the G^T-algebra $\Phi\langle a, h\rangle = (a, \mathcal{A})$ for $\langle a, h\rangle \in \mathcal{E}ns^T$, we may confine ourselves to natural transformations $\theta : (G^T)^n \xrightarrow{\cdot} G^T$ because any $k \in \mathcal{E}ns$ is a coproduct in $\mathcal{E}ns$ of one-element sets. Then $\mathcal{A}_{n,1}(\theta) = h\sigma_a$, where $\sigma : (Id_{\mathcal{E}ns})^n \xrightarrow{\cdot} T$ is a unique natural transformation from the definition of a right Kan extension $\varepsilon : TG^T \xrightarrow{\cdot}$ $\xrightarrow{\cdot} G^T$ (it follows from [12],Th.9.3, compare with [15],p. 111).

Let Z be a full subcategory of some equational category V-$\mathcal{A}lg$. We define rank Z to be the least cardinal number $\varkappa$ with the property: If $(a, \mathcal{A}), (b, \mathcal{B}) \in Z$ and $f : a \to b$ in $\mathcal{E}ns$ such that the diagram

$$\begin{array}{ccc} a^n & \xrightarrow{\quad f^n \quad} & b^n \\ {\scriptstyle \mathcal{A}_{n,k}(\theta)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{B}_{n,k}(\theta)} \\ a^k & \xrightarrow{\quad f^k \quad} & b^k \end{array}$$

commutes for each $\theta : V^n \xrightarrow{\cdot} V^k$, $card\ n < \varkappa$, then f is a V-homomorphism. Any varietal category with a rank is a full subcategory of some $\mathcal{A}(\Delta)$.

§ 2. F-strong and F-nice embeddings.

At first, we give another way in which nice embeddings can be introduced. Let $F : A \longrightarrow B$ be a functor and (N, W) a category structured over B. Let N_F be a category with objects (a, m), where $a \in A$, $m \in N$, $Wm = Fa$ and arrows $f : (a, m) \longrightarrow (a', m')$ are those arrows $f : a \longrightarrow a'$ for which $Ff = Wf'$ for some $f' : m \longrightarrow m'$. Define $W_F : N_F \longrightarrow A$ by $W_F(a, m) = a$, $W_F(f) = f$. Clearly (N_F, W_F) is structured over A (these categories were introduced in a special case in [9], 1.1). Now, let (M, U) be structured over A. It can be easy to see that F-nice embeddings $M \longrightarrow N$ are precisely realizations $M \longrightarrow N_F$. Namely, if $H : M \longrightarrow N$ is F-nice, then $m \longmapsto (Um, Hm)$ defines a realization $M \longrightarrow N_F$ and conversely, if $Gm = (a, m)$ for a realization $G : M \longrightarrow N_F$, then $m \longmapsto m$ defines an F-nice embedding $M \longrightarrow N$.

Theorem 1. Let (M, U) be structured over A and $F : A \longrightarrow B$ a functor. Let there exist an F-strong (F-nice) embedding H into a category $V\text{-}Alg$ for some $V : X \longrightarrow B$. Then $\Phi_{FU} : M \longrightarrow FU\text{-}Alg$ is an F-strong (F-nice) embedding.

If $B = Ens$ and H is F-strong, then rank $\Phi_{FU} M \leq$ rank HM.

Proof. Since $\|_{FU} \Phi_{FU} = FU = \|_V H$, the functor Φ_{FU} is faithful. Let $Hm = (FUm, \mathcal{L}^m)$ for $m \in M$. Let $n, k \in B$, $\theta : V^n \xrightarrow{\cdot} V^k$. The diagram

$$\begin{array}{ccc} (FUm)^n & \xrightarrow{\ (FUf)^n\ } & (FUm')^n \\ \Big\downarrow {\scriptstyle \mathcal{L}^m_{n,k}(\theta)} & & \Big\downarrow {\scriptstyle \mathcal{L}^{m'}_{n,k}(\theta)} \\ (FUm)^k & \xrightarrow{\ (FUf)^k\ } & (FUm')^k \end{array}$$

commutes for any $m, m' \in M$, $f : m \longrightarrow m'$ in M. Hence $\theta^*_m = \mathcal{L}^m_{n,k}(\theta)$ determines a natural transformation θ^*: $: (FU)^n \overset{\cdot}{\longrightarrow} (FU)^k$. It is $\Phi_{FU} m = (FUm, \mathcal{A}^m)$, where $\mathcal{A}^m_{n,k}(\psi) = \psi_m$ for any $\psi : (FU)^n \overset{\cdot}{\longrightarrow} (FU)^k$, i.e. $\mathcal{A}^m_{n,k}(\theta^*) = \theta^*_m = \mathcal{L}^m_{n,k}(\theta)$. Let $m, m' \in M$, $m \neq m'$. Since $Hm \neq Hm'$, there exist $n, k \in B$, $\theta : V^n \overset{\cdot}{\longrightarrow} V^k$ with $\mathcal{L}^m_{n,k}(\theta) \neq \mathcal{L}^{m'}_{n,k}(\theta)$. Hence $\mathcal{A}^m_{n,k}(\theta^*) \neq \mathcal{A}^{m'}_{n,k}(\theta^*)$ and therefore $\Phi_{FU} m \neq \Phi_{FU} m'$. We have proved that Φ_{FU} is an embedding and $|\ |_{FU}\, \Phi_{FU} = FU$.

Let $m, m' \in M$, $h : \Phi_{FU} m \longrightarrow \Phi_{FU} m'$ in $FU\text{-}Alg$ ($|\ |_{FU}(h) = F(h_1)$ for some $h_1 : Um \longrightarrow Um'$ in A in the case of an F-nice embedding). Let $n, k \in B$ and $\theta : V^n \overset{\cdot}{\longrightarrow} V^k$. It holds $h^k \cdot \mathcal{L}^m_{n,k}(\theta) = h^k \cdot \mathcal{A}^m_{n,k}(\theta^*) = \mathcal{A}^{m'}_{n,k}(\theta^*) \cdot h^n = \mathcal{L}^{m'}_{n,k}(\theta) \cdot h^n$

and therefore $h : Hm \longrightarrow Hm'$ is a V-homomorphism. Hence there exists $h' : m \longrightarrow m'$ in M with $Hh' = h$. Clearly $\Phi_{FU}(h') = h$.

Suppose $B = Ens$, H F-strong and $\varkappa = \operatorname{rank} HM$. Let $m, m' \in M$, $h : FUm \longrightarrow FUm'$ in Ens such that the diagram

$$\begin{array}{ccc}
(FUm)^{n} & \xrightarrow{\ \mathfrak{h}^{n}\ } & (FUm')^{n} \\
\Big\downarrow {\scriptstyle \mathfrak{A}^{m}_{n,\mathfrak{k}}(\psi)} & & \Big\downarrow {\scriptstyle \mathfrak{A}^{m'}_{n,\mathfrak{k}}(\psi)} \\
(FUm)^{\mathfrak{k}} & \xrightarrow{\ \mathfrak{h}^{\mathfrak{k}}\ } & (FUm')^{\mathfrak{k}}
\end{array}$$

commutes for each $n, \mathfrak{k} \in Ens$, $card\ n < \varkappa$, $\psi : (FU)^{n} \dot{\longrightarrow} (FU)^{\mathfrak{k}}$.

Hence $\mathfrak{h}^{\mathfrak{k}} \cdot \mathfrak{B}^{m}_{n,\mathfrak{k}}(\theta) = \mathfrak{B}^{m'}_{n,\mathfrak{k}}(\theta) . \mathfrak{h}^{n}$ for each $n, \mathfrak{k} \in Ens$, $card\ n < \varkappa$, $\theta : V^{n} \dot{\longrightarrow} V^{\mathfrak{k}}$. By the definition of a rank one gets that $\mathfrak{h} : Hm \longrightarrow Hm'$ is a V-homomorphism. Therefore $\mathfrak{h} = H\mathfrak{h}'$ for some $\mathfrak{h}' : m \longrightarrow m'$ in M and $\mathfrak{h} = \Phi_{FU}(\mathfrak{h}') : \Phi_{FU} m \longrightarrow \Phi_{FU} m'$ is an arrow in FU-Alg. Hence rank $\Phi_{FU} M \leq \varkappa$.

Corollary 1. Let (M, U) be structured over A, $F : A \longrightarrow B$ and FU admit a codensity monad R_{FU}. Let there exist an F-strong (F-nice) embedding H into a category V-Alg for some $V : X \longrightarrow B$. Then $\overline{FU} : M \longrightarrow B^{R_{FU}}$ is an F-strong (F-nice) embedding.

If $B = Ens$ and H is F-strong, then rank $\overline{FU} M \leq$ rank HM.

This corollary follows from Theorem 1 and from the above quoted Theorem 9.3 of [12]. We shall give an independent proof for the case V-$Alg = B^{T}$, where T is a monad in B. Let $Hm = \langle FUm, \mathfrak{h}_{m} \rangle$ for $m \in M$. Since $H : M \longrightarrow B^{T}$ is a functor, $\mathfrak{h} : TFU \dot{\longrightarrow} FU$ is a natural transformation.

Hence there exists a unique natural transformation $\sigma : T \xrightarrow{\cdot} R_{FU}$ with $h = \varepsilon \cdot \sigma FU$. Let $f : \overline{FU}m \quad \overline{FU}m'$ be an arrow in $B^{R_{FU}}$. Consider the following diagram

$$\begin{array}{ccc}
FUm & \xrightarrow{\quad f \quad} & FUm' \\
\uparrow{\scriptstyle \varepsilon_m} & & \uparrow{\scriptstyle \varepsilon_{m'}} \\
R_{FU}FUm & \xrightarrow{\quad R_{FU}f \quad} & R_{FU}FUm' \\
\uparrow{\scriptstyle \sigma_{FUm}} & & \uparrow{\scriptstyle \sigma_{FUm'}} \\
TFUm & \xrightarrow{\quad Tf \quad} & TFUm'
\end{array}$$

Since $f : \langle FUm, \varepsilon_m \rangle \longrightarrow \langle FUm', \varepsilon_{m'} \rangle$ is a homomorphism and $\sigma : T \xrightarrow{\cdot} R_{FU}$, both squares of this diagram commute. Hence $f : Hm \longrightarrow Hm'$ is a homomorphism. This fact is sufficient for the proof.

The assertion about a rank does not hold for F-nice embeddings as follows from Theorem 2. Further, this machinery does not work for full embeddings as we can see from the example of the category of ordered sets which is fully embeddable into a category of algebras $\mathfrak{A}(\Delta)$ (by [7] because a two-element chain forms a dense, i.e. left adequate in the sense of Isbell, subcategory) and Id_{Ens} is a codensity monad of its forgetful functor. In the case $A = B = Ens$ and $F = Id_{Ens}$ we obtain a necessary and sufficient condition for realizability of a concrete category (M, U) into an equational category. Moreover, the image of M in this realization has the smallest possible rank. Hence no equational category can be realized in an equational category with a smaller rank.

Corollary 2. Let $F: \mathbf{Ens} \longrightarrow \mathbf{Ens}$ be a functor. A small concrete category $(\mathcal{M}, \mathcal{U})$ which is F-strongly (F-nicely) embeddable into an equational category is F-strongly (F-nicely) embeddable into some $\mathcal{O}\mathcal{C}(\Delta)$.

Proof. By Corollary 1 $\overline{F\mathcal{U}}: \mathcal{M} \longrightarrow \mathbf{Ens}^{R_{F\mathcal{U}}}$ is an F-strong (F-nice) embedding. Let $\varkappa = \sup\{\operatorname{card} \mathcal{U}m / m \in \mathcal{M}\}$. By [15], p.112 rank $\overline{F\mathcal{U}}\mathcal{M} \leq \varkappa$. Hence $\overline{F\mathcal{U}}\mathcal{M}$ is realizable into a category of algebras endowed with a set of at most $\varkappa$-ary operations.

Let $\mathcal{M}$ have the only one object m. Then $C = \mathcal{M}(m,m)$ is a semigroup of transformations of a set $x = \mathcal{U}m$. We can compute the codensity monad $R_{\mathcal{U}}$ and we obtain that $R_{\mathcal{U}}x =$

$$= \operatorname{Lim}((x \downarrow \mathcal{U}) \xrightarrow{Q} \mathcal{M} \xrightarrow{\mathcal{U}} \mathbf{Ens}) = \{(y_f)_{f \in x^x} \in \prod_{f \in x^x} x / h(y_f) = y_{hf}$$

for any $h \in C\}$. Further $g: x \longrightarrow x$ is a homomorphism of an $R_{\mathcal{U}}$-algebra $\langle x, \varepsilon_m \rangle$ if and only if $g(y_{id_x}) = y_g$.

We have obtained a characterization of semigroups C of transformations of a set x which are endomorphism semigroups of a V-algebra as semigroups C containing id_x with the property:

$g: x \rightarrow x,\ g(y_{id_x}) = y_g$ for each $(y_f)_{f \in x^x} \in \prod_{f \in x^x} x$ with $h(y_f) = y_{hf}$ for any $h \in C \Longrightarrow g \in C$.

It was proved in [4] that a semigroup $C \subseteq X^X$ containing id_X is an endomorphism semigroup of an algebra with infinitary operations if and only if $Z(Z(L_X)) = L_X$, where Z denotes the centralizer and L_X is the family of all left translations of X^X induced by elements of C . Of course, both characterizations are equivalent.

Let $F, G : A \longrightarrow A$ be functors. Define a category $(A(F,G), U)$ structured over A as follows (see [20] for $A = \mathbf{Ens}$). The objects are couples $(a, \varkappa)$, where $a \in A$ and $\varkappa : Fa \longrightarrow Ga$ is an arrow in A . The arrows $f : (a,\varkappa) \longrightarrow (a', \varkappa')$ are arrows $f : a \longrightarrow a'$ of A such that $G(f)\varkappa = \varkappa' F(f)$. Further, $U(a,\varkappa) = a$ and $Uf = f$. If T is a monad in A , the category A^T is a full subcategory of $A(T, Id_A)$.

<u>Theorem 2</u>. Let $\langle T, \eta, \mu \rangle$ be a monad in A . Then A^T is T -nicely embeddable into $A(Id_A, Id_A)$.

<u>Proof</u>. The assignment

$$\begin{array}{ccc} \langle a, h \rangle & \longmapsto & (Ta, \eta_a h) \\ \downarrow f & & \downarrow Tf \\ \langle a', h' \rangle & \longmapsto & (Ta', \eta_{a'} h') \end{array}$$

defines a functor $H : A^T \longrightarrow A(Id_A, Id_A)$ and $TG^T = UH$ holds. Let $f, g : \langle a, h \rangle \longrightarrow \langle a', h' \rangle$ be T -homomorphisms and $Tf = Tg$. Since $h \cdot \eta_a = id_a$ by the definition of a T -algebra, it holds $f = fh.\eta_a = h' T(f) \eta_a = h' T(g) \eta_a = g h \eta_a = g$ and thus H is faithful. Let $f : a \longrightarrow a'$ be an arrow in A and $Tf : H\langle a, h \rangle \longrightarrow H\langle a', h' \rangle$ an arrow in

$\mathbf{A}(Id_A, Id_A)$. We have $h'T(f) = (h'\eta_{a'})h'T(f) = h'(\eta_{a'}, h'T(f)) = $ $= h'(T(f)\eta_a h) = h'(\eta_{a'} f h) = fh$ and thus f is a T-homomorphism. We have proved that H is a T-nice embedding.

For $\mathbf{A} = \mathcal{E}ns$ this result follows from [19], Prop.3.11, too. By this theorem any varietal category is nicely embeddable into the category of algebras with one unary operation.

Lemma 1. Let $\mathbf{A}$ have countable copowers. Then $\mathbf{A}(Id_A, Id_A)$ is monadic.

Proof. We are going to show that the forgetful functor $U: \mathbf{A}(Id, Id) \longrightarrow \mathbf{A}$ has a left adjoint F. Let $a \in \mathbf{A}$. Define $Fa = (UFa, \varkappa_a)$, where UF_a is the coproduct of countable many copies of a with injections $i_k^a : a_k = a \longrightarrow UFa$ for $k = 1, 2, \dots$ and $\varkappa_a : UFa \longrightarrow UFa$ is a unique arrow in $\mathbf{A}$ such that $i_{k+1}^a = \varkappa_a i_k^a$ for any $k = 1, 2, \dots$. If $f: a \longrightarrow b$ is an arrow in $\mathbf{A}$, then UFf is a unique arrow such that $i_k^b f = UF(f) i_k^a$ for any $k = 1, 2, \dots$. The following computation proves UFf to be an arrow in $\mathbf{A}(Id, Id)$; $UF(f).\varkappa_a . i_k^a = UF(f). i_{k+1}^a = i_{k+1}^b . f = \varkappa_b . i_k^b . f = $ $= \varkappa_b . UF(f). i_k^a$ for any k and thus $UF(f)\varkappa_a = \varkappa_b UF(f)$. Further, the equality $\eta_a = i_1^a$ defines a natural transformation $\eta : Id_A \dot{\longrightarrow} UF$. For any $x = (Ux, q) \in \mathbf{A}(Id, Id)$ there exists a unique arrow $U e_x : UFUx \longrightarrow Ux$ such that $q^{k-1} = Ue_x . i_k^{Ux}$ ($q^0 = id_{Ux}$, $q^1 = q$, $q^2 = qq$, and so on) for any $k = 1, 2, \dots$.

Moreover, $e_x : (UFUx, \varkappa_{Ux}) \longrightarrow (Ux, q)$ is an arrow in

$A(Id, Id)$ because $U\varepsilon_x . \varkappa_{Ux} . i_k^{Ux} = U\varepsilon_x . i_{k+1}^{Ux} = g^k = g . g^{k-1} =$

$= g . U\varepsilon_x . i_k^{Ux}$ for any k. We compute that $\varepsilon : FU \dot{\longrightarrow} Id_{A(Id,Id)}$ is a natural transformation. Namely, for any arrow $f : (Ux, g) \longrightarrow (Uy, h)$ in $A(Id, Id)$ it holds $Uf . U\varepsilon_x . i_k^{Ux} = Uf . g^{k-1} = h^{k-1} . Uf = U\varepsilon_y . i_k^{Uy} . Uf =$

$= U\varepsilon_y . UFUf . i_k^{Ux}$. Since $U\varepsilon_x . \eta_{Ux} = U\varepsilon_x . i_1^{Ux} =$

$= g^0 = id_{Ux}$ for any $x \in A(Id, Id)$, $U\varepsilon . \eta U : U \dot{\longrightarrow} U$ is the identity natural transformation. Let $a \in A$. It holds $U\varepsilon_{Fa} . UF\eta_a . i_k^a = U\varepsilon_{Fa} . UFi_1^a . i_k^a = U\varepsilon_{Fa} . i_k^{UFa} . i_1^a = \varkappa_a^{k-1} . i_1^a = i_k^a$

and therefore $\varepsilon F . F\eta : F \dot{\longrightarrow} F$ is the identity natural transformation, too. We have proved that F is a left adjoint for U with the unit η and counit ε.

By the Beck's precise tripleability theorem it remains to establish that U creates split coequalizers. But U creates all coequalizers. Namely, let $f, g : (a, \varkappa) \longrightarrow (a', \varkappa')$ be two arrows in $A(Id, Id)$ and $e : a' \longrightarrow a''$ a coequalizer of Uf, Ug in A. There exists a unique $\varkappa'' : a'' \longrightarrow a''$ in A such that $\varkappa'' . e = e . \varkappa'$. It is routine to prove that $e : (a', \varkappa') \longrightarrow (a'', \varkappa'')$ is a coequalizer of f and g in $A(Id, Id)$.

Corollary 3. Let A have countable copowers. Then any category comonadic over A can be nicely embedded into a category monadic over A.

The proof follows from the dual of Theorem 2 and from

Lemma 1.

The second part of this paper will appear in this journal later.

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