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KAČANOV - GALERKIN METHOD x)

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Abstract: In our previous paper it is proved the convergence of approximants (obtained by the Kačanov's method) of the minimum of nonquadratic functional. In this note, we extend this result on the Kačanov-Galerkin method.

Key words: Minimum of a nonlinear functional, Kačanov-Galerkin method.

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1. Statement of results.

Let $\tilde{H}$ be a Hilbert space with the inner product $(\cdot,\cdot)$ and let H be a closed subspace of $\tilde{H}$ with the same inner product. Suppose that $f: \tilde{H} \rightarrow R_1$ is a functional (nonquadratical, generally) defined on $\tilde{H}$ and with the Gâteaux derivative $f'(u)$ in each point $u \in \tilde{H}$ such that $f': \tilde{H} \rightarrow \tilde{H}$ is a continuous mapping which takes the bounded subsets of $\tilde{H}$ onto the bounded subsets. Let $\varphi \in \tilde{H}$ and $x^* \in \tilde{H}$.

Let Φ be a functional defined on $\tilde{H} \times \tilde{H} \times \tilde{H}$ such that

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x) The communication of authors on the conference "On Basic Problems in Numerical Analysis" held in Prague, August 27-31, 1973, dealt with such problems. Because it will be not possible to include the proof of the main result in the Proceedings of the said Conference, we present this here.

$\Phi(u,\cdot,\cdot): \widetilde{H} \times \widetilde{H} \to R_1$ is a bilinear and symmetric form on $\widetilde{H} \times \widetilde{H}$ for each fixed $u \in \widetilde{H}$.

Let c_1, c_2, c_3 be positive numbers. Suppose that for each $h \in H$ and $u, v, w \in \widetilde{H}$ the following conditions are fulfilled:

(i) $(h, f'(u+h) - f'(u)) \geq c_1 \|h\|^2$,

(ii) $\Phi(u,h,h) \geq c_2 \|h\|^2$,

(iii) $\Phi(u,u,h) = (h, f'(u))$,

(iv) $1/2\, \Phi(u,v,v) - 1/2\, \Phi(u,u,u) - f(v) + f(u) \geq 0$,

(v) $\Phi(u,v,w) \leq c_3 \|v\| \cdot \|w\|$.

From the well-known theorem (see e.g. [4, Theorem 9.2]) and from the assumption (i) it follows that there exists a uniquely determined $x_0 \in H$ satisfying

(1) $f(x_0 + x^*) - (x_0 + x^*, \varphi) = \min_{v \in H} \{f(v + x^*) - (v + x^*, \varphi)\}$.

Let $\{\varphi_n\}_{n=1}^{\infty} \subset \widetilde{H}$, $\{x_n^*\}_{n=1}^{\infty} \subset \widetilde{H}$, $\varphi_n \to \varphi$ (the convergence in the norm of the space $\widetilde{H}$), $x_n^* \to x^*$. Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of the closed subspaces of the space H such that

(vi) $H_n \subset H_{n+1}$ $(n = 1, 2, \ldots)$, $\overline{\bigcup H_n} = H$.

Let $x_1 \in H_1$. Then (again by [4, Theorem 9.2]) there exists a uniquely determined sequence $\{x_n\}_{n=1}^{\infty} \subset H$, such that $x_n \in H_n$ and

(2) $1/2\, \Phi(x_n + x_{n+1}^*, x_{n+1} + x_{n+1}^*, x_{n+1} + x_{n+1}^*) -$

$$- (x_{n+1} + x_{n+1}^*, \varphi_{n+1}) =$$

$$= \min_{v \in H_{n+1}} \{1/2\, \Phi(x_n + x^*_{n+1}, v + x^*_{n+1}, v + x^*_{n+1}) -$$
$$- (v + x^*_{n+1}, \varphi_{n+1})\} \qquad (n = 1, 2, \dots) .$$

<u>Convergence Theorem</u>. Under the assumptions (i) - (vi) and if the series

$$\sum_{n=1}^{\infty} \| x^*_{n+1} - x^*_n \| , \quad \sum_{n=1}^{\infty} \| \varphi_{n+1} - \varphi_n \|$$

are convergent, then

$$\lim_{n \to \infty} \| x_n - x_0 \| = 0 .$$

2. <u>Proof of Convergence Theorem.</u>

For each $v \in \tilde{H}$ and arbitrary positive integer n put

$$\mathcal{T}_n(v) = f(x_n + x^*_{n+1}) - (v + x^*_{n+1}, \varphi_{n+1}) +$$
$$+ 1/2\, \Phi(x_n + x^*_{n+1}, v + x^*_{n+1}, v + x^*_{n+1}) -$$
$$- 1/2\, \Phi(x_n + x^*_{n+1}, x_n + x^*_{n+1}, x_n + x^*_{n+1}) ,$$
$$\psi_n(v) = f(v + x^*_{n+1}) - (v + x^*_{n+1}, \varphi_{n+1}) ,$$
$$F_{n+1}(v) = 1/2\, \Phi(x_n + x^*_{n+1}, v + x^*_{n+1}, v + x^*_{n+1}) -$$
$$- (v + x^*_{n+1}, \varphi_{n+1}) .$$

<u>Lemma 1</u>. For any $h \in H_{n+1}$ and $n = 1, 2, \dots$ it is

$$\Phi(x_n + x^*_{n+1}, x_{n+1} + x^*_{n+1}, h) = (h, \varphi_{n+1}) .$$

<u>Proof</u>. The functional F_{n+1} attains at the point x_{n+1} the minimum on the space H_{n+1}, i.e.

$$F_{n+1}(x_{n+1}) = \min_{v \in H_{n+1}} F_{n+1}(v) .$$

Thus the Gâteaux derivative at the point x_{n+1} must vanish, i.e. for any $h \in H_{n+1}$ we have

$$0 = DF_{n+1}(x_{n+1}, h) = \Phi(x_n + x^*_{n+1}, x_{n+1} + x^*_{n+1}, h) - (h, \varphi_{n+1})$$

and from this follows our assertion.

<u>Lemma 2</u>. The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

<u>Proof</u>. From the assumptions (ii) and (v) we have:

$$c_2 \| x_{n+1} \|^2 \leq \Phi(x_n + x^*_{n+1}, x_{n+1}, x_{n+1}) =$$
$$= \Phi(x_n + x^*_{n+1}, x_{n+1} + x^*_{n+1}, x_{n+1}) -$$
$$- \Phi(x_n + x^*_{n+1}, x^*_{n+1}, x_{n+1}) = (x_{n+1}, \varphi_{n+1}) -$$
$$- \Phi(x_n + x^*_{n+1}, x^*_{n+1}, x_{n+1}) \leq \| x_{n+1} \| \cdot \| \varphi_{n+1} \| +$$
$$+ c_3 \| x^*_{n+1} \| \, \| x_{n+1} \| .$$

Since the sequences $\{\| \varphi_n \|\}_{n=1}^{\infty}$ and $\{\| x^*_n \|\}_{n=1}^{\infty}$ are bounded, the last inequalities imply our assertion.

<u>Lemma 3</u>. $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$.

<u>Proof</u>. From the relation

$$F_{n+1}(x_{n+1}) = \min_{v \in H_{n+1}} F_{n+1}(v) \leq F_{n+1}(x_n)$$

we obtain

$$1/2\, \Phi(x_n + x^*_{n+1}, x_{n+1} + x^*_{n+1}, x_{n+1} + x^*_{n+1}) -$$
$$- (x_{n+1} + x^*_{n+1}, \varphi_{n+1}) \leq$$
$$\leq 1/2\, \Phi(x_n + x^*_{n+1}, x_n + x^*_{n+1}, x_n + x^*_{n+1}) - (x_n + x^*_{n+1}, \varphi_{n+1})$$

and

$$\pi_n(x_{n+1}) \leq f(x_n + x^*_{n+1}) - (x_n + x^*_{n+1}, \varphi_{n+1}) = \psi_n(x_n) .$$

Since

$$\pi_n(x_{n+1}) - \psi_n(x_{n+1}) = f(x_n + x^*_{n+1}) +$$
$$+ 1/2\, \Phi(x_n + x^*_{n+1}, x_{n+1} + x^*_{n+1}, x_{n+1} + x^*_{n+1}) -$$
$$- 1/2\, \Phi(x_n + x^*_{n+1}, x_n + x^*_{n+1}, x_n + x^*_{n+1}) - f(x_{n+1} + x^*_{n+1})$$

we obtain (using the assumption (iv))

$$(3) \qquad \psi_n(x_{n+1}) \leq \pi_n(x_{n+1}) \leq \psi_n(x_n) \qquad (n = 1, 2, \dots) .$$

Under our assumptions the functional f satisfies the Lipschitz condition on any bounded subset of the space $\tilde{H}$ and in virtue of Lemma 2 there exists a constant $K > 0$ such that

$$(4) \qquad \psi_n(x_{n+1}) \geq \psi_{n+1}(x_{n+1}) - K(\|x^*_{n+1} - x^*_{n+2}\| + \|\varphi_{n+1} - \varphi_{n+2}\|)$$

for $n = 1, 2, \dots$. Put

$$\varepsilon_n = K(\|x^*_{n+1} - x^*_{n+2}\| + \|\varphi_{n+1} - \varphi_{n+2}\|)$$

and

$$\vartheta_n = \psi_n(x_n) - \sum_{j=1}^{n-1} \varepsilon_j .$$

From (3) and (4) it follows

$$(5) \qquad \vartheta_1 \geq \vartheta_2 \geq \dots \geq \vartheta_n \geq \vartheta_{n+1} \geq \dots .$$

So using the assumption (ii),(3),(4) and Lemma 1, we have

$$(6) \quad c_2\|x_{n+1} - x_n\|^2 \leq \Phi(x_n + x^*_{n+1}, x_{n+1} - x_n, x_{n+1} - x_n) =$$

$$= \Phi(x_n + x^*_{n+1}, x_{n+1} + x^*_{n+1}, x_{n+1} - x_n) -$$
$$- \Phi(x_n + x^*_{n+1}, x_n + x^*_{n+1}, x_{n+1} - x_n) =$$
$$= (x_{n+1} - x_n, \varphi_{n+1}) - \Phi(x_n + x^*_{n+1}, x_n + x^*_{n+1}, x_{n+1} - x_n) \leq$$
$$\leq 2(\psi_n(x_n) - \pi_n(x_{n+1})) \leq$$
$$\leq 2(\psi_n(x_n) - \psi_{n+1}(x_{n+1}) + \varepsilon_n) = 2(\vartheta_n - \vartheta_{n+1}) .$$

The sequence $\{\vartheta_n\}_{n=1}^{\infty}$ is bounded and with respect to (5) we have

$$\lim_{n\to\infty} (\vartheta_n - \vartheta_{n+1}) = 0$$

and from this and from (6) our assertion follows.

<u>Lemma 4</u>. Let n be a fixed positive integer. Under the assumption (i) there exists a unique point $x_0^n \in H$ such that

$$f(x_0^n + x_n^*) - (x_0^n + x_n^*, \varphi_n) = \min_{v \in H} \{ f(v + x_n^*) - (v + x_n^*, \varphi_n) \} .$$

Then it is

$$\lim_{n\to\infty} \| x_0^n - x_0 \| = 0 .$$

<u>Proof</u>. The sequence $\{x_0^n\}_{n=1}^{\infty}$ is bounded. Since in the points in which the considered functionals attain their minimum, the Gâteaux derivatives are zero, we have

$$(f'(x_0^n + x_n^*), x_0^n - x_0) = (\varphi_n, x_0^n - x_0)$$

and

$$(f'(x_0 + x^*), x_0^n - x_0) = (\varphi, x_0^n - x_0) \quad (n = 1, 2, \ldots) .$$

With respect to the assumption (i) we have

$$c_1\|x_0^m - x_0\|^2 \le (f'(x_0^m + x_m^*) - f'(x_0 + x_m^*),\ x_0^m - x_0) =$$
$$= (\varphi_m - \varphi,\ x_0^m - x_0) + (f'(x_0 + x^*) - f'(x_0 + x_m^*),\ x_0^m - x_0) \le$$
$$\le \|\varphi_m - \varphi\| \cdot \|x_0^m - x_0\| + \|f'(x_0 + x^*) -$$
$$- f'(x_0 + x_m^*)\| \cdot \|x_0^m - x_0\| \ .$$

The last inequalities, continuity of the mapping f' and our assumptions imply our assertion.

Now, we are ready to finish the proof of Convergence Theorem. Let P_m be the orthogonal projection from H onto H_m. Then

$$c_1\|x_m - x_0\|^2 \le (x_m - x_0,\ f'(x_m + x_{m+1}^*) - f'(x_0 + x_{m+1}^*)) =$$
$$= (x_m - x_0,\ f'(x_m + x_{m+1}^*)) - (x_m - x_0,\ f'(x_0 + x_{m+1}^*)) =$$
$$= \Phi(x_m + x_{m+1}^*,\ x_m + x_{m+1}^*,\ x_m - x_0) -$$
$$- (x_m - x_0,\ f'(x_0 + x_{m+1}^*)) = \Phi(x_m + x_{m+1}^*,\ x_{m+1} + x_{m+1}^*,\ x_m - x_0) +$$
$$+ \Phi(x_m + x_{m+1}^*,\ x_m - x_{m+1},\ x_m - x_0) - (x_m - x_0,\ f'(x_0 + x_{m+1}^*)) =$$
$$= \Phi(x_m + x_{m+1}^*,\ x_m - x_{m+1},\ x_m - x_0) +$$
$$+ \Phi(x_m + x_{m+1}^*,\ x_{m+1} + x_{m+1}^*,\ x_m - P_m x_0) +$$
$$+ \Phi(x_m + x_{m+1}^*,\ x_{m+1} + x_{m+1}^*,\ P_m x_0 - x_0) -$$
$$- (x_m - P_m x_0,\ f'(x_0 + x_{m+1}^*)) + (x_0 - P_m x_0,\ f'(x_0 + x_{m+1}^*)) \le$$
$$\le c_3\|x_m - x_{m-1}\| \cdot \|x_m - x_0\| + (x_m - P_m x_0,\ \varphi_{m+1}) +$$
$$+ c_3\|x_{m+1} + x_{m+1}^*\| \cdot \|P_m x_0 - x_0\| +$$
$$+ \|x_0 - P_m x_0\| \cdot \|f'(x_0 - x_{m+1}^*)\| -$$
$$- (x_m - P_m x_0,\ f'(x_0^{m+1} + x_{m+1}^*)) +$$

$$+ (x_n - P_n x_0, f'(x_0^{n+1} + x_{n+1}^*) - f'(x_0 + x_{n+1}^*)) \leq$$

$$\leq c_3 \| x_n - x_{n+1} \| . \| x_n - x_0 \| + c_3 \| x_{n+1} + x_{n+1}^* \| \, \| P_n x_0 - x_0 \| +$$

$$+ \| f'(x_0 + x_{n+1}^*) \| . \| x_0 - P_n x_0 \| +$$

$$+ \| x_n - P_n x_0 \| . \| f'(x_0^{n+1} + x_{n+1}^*) - f'(x_0 + x_{n+1}^*) \| .$$

From the last inequalities, Lemma 3 and 5, continuity of the mapping f' and boundedness of the sequence $\{ \| x_n \| \}_{n=1}^{\infty}$, we obtain the desired result.

3. Remarks.

a) The main ideas of the Kačanov method are explained in the book of S.G. Michlin [3, pp.369-370].

b) The proof of the convergence of the Kačanov method is given in the authors' paper [1] where also the application to the second and mixed problems for elastoplastic materials with the using of the deformation theory of plasticity is stated.

c) The convergence of the Kačanov method in the special case for the solving of the magnetostatic field in nonlinear media has been proved in the paper of J. Kačur, J. Nečas, J. Polák and J. Souček [2].

d) Since the assumptions of our Convergence Theorem for the Kačanov-Galerkin method are essentially the same as in the Kačanov method, we can apply this method to the same problems.

R e f e r e n c e s

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