Miroslav Dont

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ON A HEAT POTENTIAL

(Preliminary communication)

Miroslav DONT, Praha

Abstract: In this note we deal with a heat potential and its boundary behaviour in connection with the Fourier problem for the heat equation in R^2 . For this purpose we define the so-called parabolic variation.

Key words: Potential, heat potential, double-layer potential, single-layer potential, parabolic variation, limits of potentials

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Let Γ be the well-known kernel in R^2 defined by

$$\Gamma(x,t) = \begin{cases} (\pi t)^{-\frac{1}{2}} \exp(-\frac{x}{4t}), & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and denote by $\partial_1 \Gamma$ its partial derivative with respect to the variable x . Fix $a < b$ in R^1 and let $C_0(\langle a, b \rangle)$ stand for the space of all continuous real-valued functions on $\langle a, b \rangle$ vanishing at a . Let φ be a fixed continuous function of bounded variation on $\langle a, b \rangle$ and put

$$K = \{[\varphi(t), t]; a \leq t \leq b\},$$

$D_K^+ = \{[x,t];\ a < t < b,\ x > \varphi(t)\}$, $D_K^- = \{[x,t];\ a < t < b,\ x < \varphi(t)\}$.

With each $f \in C_0(\langle a,b\rangle)$ we shall associate the function Tf on $R^2 - K$ defined by

$$Tf(x,t) = -\int_a^b f(\tau)\partial_1\Gamma(x-\varphi(\tau),t-\tau)\,d\tau - \\ -\int_a^b f(\tau)\Gamma(x-\varphi(\tau),t-\tau)\,d\varphi(\tau)$$

for $t > a$, $Tf(x,t) = 0$ for $t \leq a$.

Investigation of $Tf(x,t)$ (which is a combination of a double-layer and a single-layer heat potential) as $[x,t]$ approaches K is of importance in connection with the boundary value problems for the heat equation (compare [7], § 4 in chap. VI; see also [1],[2],[6] for references concerning heat potentials). Our purpose in this note is to present a simple necessary and sufficient condition on K guaranteeing the existence of the finite limits

(1) $\lim\limits_{\substack{[x,t]\to[x_0,t_0] \\ [x,t]\in D_K^+}} Tf(x,t) = T_1f(t_0)$, $\lim\limits_{\substack{[x,t]\to[x_0,t_0] \\ [x,t]\to D_K^-}} Tf(x,t) = T_2f(t_0)$

at $[x_0,t_0] \in K$ for any $f \in C_0(\langle a,b\rangle)$.

Given $[x_0,t_0] \in R^2$ and $\alpha > 0$ consider the parabola

$$P_\alpha(x_0,t_0) = \{[x,t] \in R^2;\ t_0 - t = \left(\frac{x_0-x}{2\alpha}\right)^2\}$$

and denote by $n_K(\alpha;x_0,t_0)$ the number of points in $(K - \{[x_0,t_0]\}) \cap P_\alpha(x_0,t_0)$ (we put $n_K(\alpha;x_0,t_0) = +\infty$

if the last set is infinite). Then $n_K(\alpha; x_0, t_0)$ is a Lebesgue measurable extended-real-valued function of the variable $\alpha \in (0, +\infty)$ and we may form the quantity

$$V_K(x_0, t_0) = \int_0^{\infty} e^{-\alpha^2} \, n_K(\alpha; x_0, t_0) \, d\alpha$$

to be termed the parabolic variation of K at $[x_0, t_0]$.

In connection with Tf the parabolic variation plays a role comparable with that of the so-called cyclic variation v^{φ} as introduced in [3] in connection with the investigation of double-layer logarithmic potentials. The following theorem holds.

Theorem. If at least one of the limits (1) exists for every $f \in C_0(\langle a, b \rangle)$ then there is a $\delta > 0$ such that

$$\sup_{\substack{|t-t_0| < \delta \\ t \in \langle a, b \rangle}} V_K(\varphi(t), t) < \infty . \tag{2}$$

Conversely, if (2) holds, then the finite limits (1) exist for every bounded Baire function f on $\langle a, b \rangle$ that is continuous at t_0 (and vanishes at a in case $t_0 = a$).

<u>Proof</u> of this theorem is based on the Banach theorem on variation of a continuous function and on ideas employed in [4] in connection with double-layer logarithmic potentials. The key part of the proof rests on the following lemma whose role is similar to that of Theorem 1.11 in [4].

<u>Lemma</u>. If (2) holds then there is a neighborhood U of $[\varphi(t_0), t_0]$ in R^2 such that

$$\sup_{[x,t]\in U} V_K(x,t) < +\infty .$$

If

$$(3) \qquad \sup_{t\in\langle a,b\rangle} V_K(\varphi(t),t) < +\infty$$

then $V_K(\cdot,\cdot)$ is bounded on the whole of E^2 .

Corollary. Tf is uniformly continuous on each of the domains D_K^+, D_K^- for any $f \in C_0(\langle a,b\rangle)$ if and only if (3) holds.

A modification of V_K permits to evaluate, in geometric ferms, the Fredholm radius of the operators $T_i + (-1)^i I$ (where I is the identity operator on $C_0(\langle a,b\rangle)$ and T_i are defined by (1)) and establish a general theorem on representability of the solution of the Fourier problem by means of Tf . The applied methods have been worked up in [5]. The following assertion holds.

Theorem. Let the Fredholm radius of $T_1 - I$ (which we can express in geometric terms) be greater than 1 . Put $B = K \cup \{[x,a]\,;\, x \geq \varphi(a)\}$ and let F be a continuous bounded function on B with $F(\varphi(a),a) = 0$. Then there is a unique function $f \in C_0(\langle a,b\rangle)$ such that the function

$$Tf(x,t) + \frac{1}{2\sqrt{\pi}} \int_{\varphi(a)}^{\infty} \frac{F(z,a)}{\sqrt{t-a}}\, e^{-\frac{(x-z)^2}{4(t-a)}}\, dz$$

is a solution of the Fourier problem on D_K^+ for the boundary condition F .

Analogical results may be obtained for the domain D_K^- and for domains of the form $\{[x,t]; t \in (a,b), \varphi_1(t) < x < \varphi_2(t)\}$ where φ_1, φ_2 are some continuous functions of bounded variation on $\langle a, b \rangle$ such that $\varphi_1(t) < \varphi_2(t)$ for each $t \in \langle a, b \rangle$.

Complete proofs of these results together with further details and bibliography will be included in a paper which will be published in the Czechoslovak Mathematical Journal.

The following two assertions will be proved in a paper which will be published in Časopis pro pěstování matematiky.

<u>Theorem</u>. Let $t \in (a, b\rangle$ and suppose that

$$\limsup_{z \to t-} \frac{|\varphi(t)-\varphi(z)|}{\sqrt{t-z}} < \infty .$$

Then there is a finite limit

$$\lim_{x \to \varphi(t)+} Tf(x,t)$$

(or a finite limit

$$\lim_{x \to \varphi(t)-} Tf(x,t))$$

for any function $f \in C(\langle a, b \rangle)$ if and only if

$$V_K(\varphi(t), t) < \infty .$$

<u>Proposition</u>. There is a continuous function φ of bounded variation on $\langle a, b \rangle$ such that

$$V_K(\varphi(t), t) = \infty$$

for almost every $t \in \langle a, b \rangle$ (where $K = \{[\varphi(t), t];$ $t \in \langle a, b \rangle\}$).

R e f e r e n c e s

[1] A. FRIEDMAN: Partial differential equations of parabolic type, Prentice-Hall 1964.

[2] L.I. KAMYNIN: O gladkosti teplovych potencialov V, Diff.uravnenija IV(1968),347-365.

[3] J. KRÁL: On the logarithmic potential, Comment.Math. Univ.Carolinae 3(1962),3-9

[4] J. KRÁL: On the logarithmic potential of the double distribution, Czechoslovak Math.Journal 14(89) (1964),306-321.

[5] J. KRÁL: The Fredholm radius of an operator in potential theory, Czechoslovak Math.Journal 15(90) (1965),454-473, 565-588.

[6] O.A. LADYŽENSKAJA, V.A. SOLONNIKOV, N.N. URAL'CEVA: Linejnyje i kvazilinejnyje uravnenija paraboličeskogo tipa, Moskva 1967.

[7] A.N. TICHONOV, A.A. SAMARSKIJ: Uravnenija matematičeskoj fiziki, Moskva 1966.

Matematicko-fyzikální fakulta

Karlova universita

Malostran. nám. 25

11 000 Praha 1

Československo

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