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Jiří Rosický

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FULL EMBEDDINGS WITH A GIVEN RESTRICTION

Jiří ROSICKÝ, Brno

Abstract: Let A, C be categories, $\mathcal{M}$ a full subcategory of C, $K: \mathcal{M} \longrightarrow C$ the inclusion functor and $T: \mathcal{M} \longrightarrow A$ a full and faithful functor. Denote by $\mathcal{F}_K(T)$ the category of all full and faithful functors $S: C \longrightarrow A$ with $SK = T$, arrows of which are natural transformations σ between two such functors having the property that σK is the identity natural transformation. There are studied conditions under which $\mathcal{F}_K(T)$ has an initial object. If $\mathcal{M}$ is small, cogenerates C and is dense in C, A is cocomplete and co-well-powered, this initial object exists.

Key-words: Category, faithful functor, natural transformation, initial object, realization, Kan extension.

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Let A, C be categories, $\mathcal{M}$ a subcategory of C, $K: \mathcal{M} \longrightarrow C$ the inclusion functor and $T: \mathcal{M} \longrightarrow A$ a functor. Denote by $\mathcal{C}_K(T)$ the category of all functors $S: C \longrightarrow A$ with $SK = T$, arrows of which are natural transformations σ between two such functors having the property that σK is the identity natural transformation. We shall consider some full subcategories of $\mathcal{C}_K(T)$ especially the full subcategory consisting of all full embeddings and the existence of initial or terminal objects of these subcategories. More precisely, we shall construct a functor from $\mathcal{C}_K(T)$ which turns out to be initial or terminal in $A \subseteq \mathcal{C}_K(T)$ when A is non-empty. Therefore, these considerations can help us in recognizing whether a full embedding $S : C \longrightarrow A$ extending T really exists. Further, we shall be interested in the existence of full embeddings having a right or left adjoint. Concerning concepts of the theory of categories see [4].

The Kan extensions

A left Kan extension of T along K is a pair consisting of a functor $L : C \longrightarrow A$ and a natural transformation $\eta : T \stackrel{\cdot}{\longrightarrow} LK$ such that for each pair $S : C \longrightarrow A$, $\alpha : T \stackrel{\cdot}{\longrightarrow} SK$ there is a unique natural transformation $\sigma : L \stackrel{\cdot}{\longrightarrow} S$ such that $\alpha = \sigma K \cdot \eta$. L is denoted by $Lan_K T$. In most cases L can be defined pointwise, for instance when M is small and A cocomplete. Then Lc for $c \in C$ is a colimit of the functor

$$(K \downarrow c) \xrightarrow{P} M \xrightarrow{T} A$$

where $(K \downarrow c)$ is the comma category having

objects: $\begin{array}{c} m \in M \\ \big\downarrow f \\ c \end{array}$ arrows: $\begin{array}{ccc} m & \xrightarrow{h} & m' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & c & \end{array}$

P is the projection $m \xrightarrow{f} c \longmapsto m$. $L(g)$ is a unique arrow commuting with the limiting cones for any arrow g

of C . In this case L is called a pointwise left Kan extension. If $\mathcal{M}$ is a full subcategory of $\mathcal{C}$ and the pointwise left Kan extension $Lan_K T$ exists, η can be chosen as the identity natural transformation. Detail information concerning Kan extensions can be found in [4].

The last result implies that if $\mathcal{M}$ is full, the pointwise left Kan extension $Lan_K T$ is an initial object of $\mathcal{C}_K(T)$.

Definition. A functor $F: C \longrightarrow A$ is called left $\mathcal{M}$ -faithful when to every $m \in \mathcal{M}$, $c \in C$ and every pair $f, g : m \longrightarrow c$ of parallel arrows of C the equality $F(f) = F(g)$ implies $f = g$.

Proposition 1. Let the left Kan extension $L = Lan_K T, \eta$ of T along K exist. Let there exist a left $\mathcal{M}$ -faithful functor $S : C \longrightarrow A$ and a pointwise epi natural transformation $\alpha : T \stackrel{\cdot}{\longrightarrow} SK$. Then L is left $\mathcal{M}$ -faithful. If $\mathcal{M}$ generates C, L is faithful.

Proof: Let $m \in \mathcal{M}$, $c \in C$ and $f \neq g : m \longrightarrow c$ be a parallel pair of arrows of C . There is a natural transformation $\sigma : L \stackrel{\cdot}{\longrightarrow} S$ such that $\alpha = \sigma K \cdot \eta$. Therefore $\sigma_c L(f) = S(f)\sigma_m$ and $\sigma_c L(g) = S(g)\sigma_m$. Since α_m is epi, σ_m is epi and therefore $L(f) \neq L(g)$ because S is left $\mathcal{M}$ -faithful.

Let $c, d \in C$ and $f \neq g : c \longrightarrow d$ be arrows of C . Since $\mathcal{M}$ generates C , there is an $m \in \mathcal{M}$ and an arrow $h : m \longrightarrow c$ with $fh \neq gh$. We have $L(fh) \neq L(gh)$ and therefore $L(f) \neq L(g)$.

Full embeddings

From now till the end of this paper we shall suppose that $\mathcal{M}$ is a full subcategory of $\mathcal{C}$.

Definition. A functor $F: \mathcal{C} \longrightarrow \mathcal{A}$ is called left $\mathcal{M}$ -full when to every $m \in \mathcal{M}$ and to every arrow g : $: Fm \longrightarrow Fc$ of $\mathcal{A}$, there is an arrow $f: m \longrightarrow c$ of $\mathcal{C}$ with $F(f) = g$.

Let $\mathcal{L}_K(T)$, $\mathcal{F}_K(T)$ and $\mathcal{E}_K(T)$ be the full subcategories of $\mathcal{C}_K(T)$ consisting of all left $\mathcal{M}$ -full and left $\mathcal{M}$ -faithful functors, full and faithful functors and of all full embeddings.

Lemma 1. Let $\mathcal{M}$ cogenerate $\mathcal{C}$. Let $L \in \mathcal{C}_K(T)$, $S \in \mathcal{L}_K(T)$ and $\sigma: L \overset{\cdot}{\longrightarrow} S$ be an arrow of $\mathcal{C}_K(T)$. Let $m \in \mathcal{M}$, $c \in \mathcal{C}$ and $f, g: Lm \longrightarrow Lc$ be a parallel pair of arrows of $\mathcal{A}$. The following conditions are equivalent:

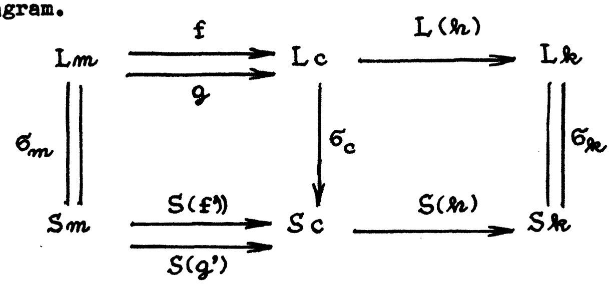
(i) $\sigma_c f = \sigma_c g$,

(ii) $L(h)f = L(h)g$ for every arrow $h: c \longrightarrow k$ of $\mathcal{C}$ and every $k \in \mathcal{M}$.

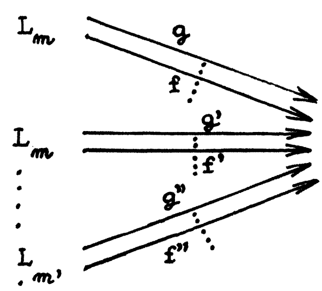
Proof: Let (i) hold, $k \in \mathcal{M}$ and $h: c \longrightarrow k$. It is $L(h) = \sigma_k L(h) = S(h)\sigma_c$. By (i) $L(h)f = L(h)g$.

Let (ii) hold and suppose that $\sigma_c f \neq \sigma_c g$. Since S is left $\mathcal{M}$ -full, there exist arrows $f', g': m \longrightarrow c$ with $\sigma_c f = S(f')$, $\sigma_c g = S(g')$. Since $f' \neq g'$ and $\mathcal{M}$ cogenerates $\mathcal{C}$, we can find a $k \in \mathcal{M}$ and an arrow $h: c \longrightarrow k$ of $\mathcal{C}$ such that $hf' \neq hg$. Hence

$S(hf') \neq S(hg')$ and therefore $S(h)\sigma_c f \neq S(h)\sigma_c g$. It implies $L(h)f \neq L(h)g$, which is a contradiction. The proof can be visualized on the following commutative diagram.



Let $L : C \longrightarrow A$ be a functor and $c \in C$. Let us have the following diagram in A .



Arrows of this diagram are all arrows of A with the domain in LM and the codomain Lc . Arrows $f, g : Lm \longrightarrow Lc$ have the same domain in this diagram if and only if $L(h)f =$ $= L(h)g$ for every arrow $h : c \longrightarrow k$ and every $k \in$ $\in m$. We denote this diagram by $D_{L,c}$.

Let M be small and C cocomplete. Let L_0 be a pointwise left Kan extension of T along K . Suppose that we have functors $L_\beta : C \longrightarrow A$ for each ordinal $\beta < \alpha$.

Let α be isolated. Define $L_\alpha c = \operatorname{Colim} D_{L_{\alpha-1}, c}$ for every $c \in C$. Let $\lambda_c^{\alpha-1,\alpha}$ be the component of the limiting cone with the domain $L_{\alpha-1} c$. Let $\varkappa : c \to c'$ and $h : c' \to k$, where $c, c' \in C$, $k \in M$. Let $f, g : L_{\alpha-1} m \to L_{\alpha-1} c$ be a parallel pair of arrows of $D_{L_{\alpha-1}, c}$. It is $L_{\alpha-1}(h) L_{\alpha-1}(\varkappa) f = L_{\alpha-1}(h\varkappa) f = L_{\alpha-1}(h\varkappa) g = L_{\alpha-1}(h) L_{\alpha-1}(\varkappa) g$ and therefore $\lambda_{c'}^{\alpha-1,\alpha} L_{\alpha-1}(\varkappa) f = \lambda_{c'}^{\alpha-1,\alpha} L_{\alpha-1}(\varkappa) g$. Hence $\lambda_{c'}^{\alpha-1,\alpha} L_{\alpha-1}(\varkappa)$ determines a cone from $D_{L_{\alpha-1}, c}$. Let $L_\alpha(\varkappa) : L_\alpha c \to L_\alpha c'$ be a unique arrow of A with $L_\alpha(\varkappa) \lambda_c^{\alpha-1,\alpha} = \lambda_{c'}^{\alpha-1,\alpha} L_{\alpha-1}(\varkappa)$. Then $L_\alpha : C \to A$ is a functor and $\lambda^{\alpha-1,\alpha} : L_{\alpha-1} \xrightarrow{\cdot} L_\alpha$ a natural transformation.

Let α be limit. Let $L_\alpha(c)$ be a colimit of the diagram having objects $L_\beta c$ and arrows $\lambda_c^{\beta,\beta+1}$ for $\beta < \alpha$ with the limiting cone $\{\lambda_c^{\beta,\alpha} : L_\beta c \to L_\alpha c\}$. Each arrow $\varkappa : c \to c'$ of C induces a unique arrow $L_\alpha(\varkappa) : L_\alpha c \to L_\alpha c'$ commuting with the limiting cones. Hence $L_\alpha : C \to A$ is a functor and $\lambda^{\beta,\alpha} : L_\beta \xrightarrow{\cdot} L_\alpha$ a natural transformation for any $\beta < \alpha$.

In both cases if $L_\alpha c$ is isomorphic to some $L_\beta c$, $\beta < \alpha$ we choose $L_\alpha c$ to be equal to this $L_\beta c$.

<u>Lemma 2.</u> $L_\alpha \in \mathcal{C}_K(T)$ for any α and for any $\beta < \alpha$ there exists a pointwise epi natural transformation $\lambda^{\beta,\alpha} : L_\beta \xrightarrow{\cdot} L_\alpha$ which is an arrow of $\mathcal{C}_K(T)$. For any $F \in \mathcal{C}_K(T)$ and for any ordinal α there is at

most one arrow $\sigma: L_\alpha \xrightarrow{\cdot} F$ of $\mathcal{C}_K(T)$.

Proof: Since $\mathcal{M}$ is a full subcategory of $\mathcal{C}$, $L_0 K =$ $= T$. Clearly $D_{L_0, m}$ has no parallel arrows for any $m \in$ $\in \mathcal{M}$. Therefore $L_1 m = L_0 m$, i.e. $L_1 K = T$. Hence $L_\alpha K = T$ for any α . Clearly $\lambda^{\beta,\alpha}$ exists for any $\beta < \alpha$ and $\lambda^{\beta,\alpha}\lambda^{\gamma,\beta} = \lambda^{\gamma,\alpha}$ for any $\gamma < \beta < \alpha$. By the transfinite induction it can be easily shown that $\lambda_c^{\beta,\alpha}$ is epi for any $\beta < \alpha$, $c \in \mathcal{C}$.

Let $F \in \mathcal{C}_K(T)$ and $\sigma, \sigma': L_\alpha \xrightarrow{\cdot} F$ be arrows of $\mathcal{C}_K(T)$. Since L_0 is a left Kan extension, we have $\sigma\lambda^{0,\alpha} = \sigma'\lambda^{0,\alpha}$. Therefore $\sigma = \sigma'$ because $\lambda^{0,\alpha}$ is pointwise epi.

Let $L_\gamma c = L_\beta c$ for some $\gamma < \beta$. Then $L_{\gamma+1} c =$ $= L_{\beta+1} c$. By Lemma 2 $\lambda_c^{\gamma+1,\beta}\lambda^{\gamma,\gamma+1} = 1_{L_\gamma c}$ and $\lambda_c^{\gamma,\gamma+1}\lambda_c^{\gamma+1,\beta} = \lambda_c^{\beta,\beta+1}\lambda^{\gamma+1,\beta} = 1_{L_{\gamma+1} c}$ and therefore $L_\gamma c = L_{\gamma+1} c$. Thus in this case $L_\gamma c = L_\delta c$ for any $\gamma \leq \delta$. Suppose that for every $c \in \mathcal{C}$ there exists an ordinal $\gamma(c)$ such that $L_{\gamma(c)} c = L_\beta c$ for any $\beta \geq \gamma(c)$. Put $L_* c = L_{\gamma(c)} c$. Let $\kappa : c \longrightarrow c'$. Suppose that $\gamma(c') \leq \gamma(c)$. Put $L_*(\kappa) = L_{\gamma(c)}(\kappa): L_* c \longrightarrow L_* c'$. In this way we obtain a functor $L_* \in \mathcal{C}_K(T)$. Let $\lambda_c^\alpha : L_\alpha c \longrightarrow L_* c$ be equal to $\lambda_c^{\alpha,\gamma(c)}$ for $\alpha < \gamma(c)$ and to the identity for $\alpha \geq \gamma(c)$. Clearly $\lambda^\alpha : L_\alpha \xrightarrow{\cdot} L_*$ is an arrow of $\mathcal{C}_K(T)$.

Proposition 2. Let $\mathcal{M}$ be small and cogenerate $\mathcal{C}$. Let A be cocomplete and co-well-powered. Let $\mathcal{L}_K(T) \neq \emptyset$.

Then L_* is an initial object in $\mathcal{L}_K(T)$.

<u>Proof</u>: Let $c \in C$. Since A is co-well-powered and any $\lambda_c^{0,\alpha}$ is epi, there exists $\gamma(c)$. Therefore, L_* is defined.

Let $S \in \mathcal{L}_K(T)$. There exists a unique natural transformation $\sigma^0 : L_0 \stackrel{\cdot}{\longrightarrow} S$ such that $\sigma^0 K$ is the identity. Suppose that such σ^β exists for each $\beta < \alpha$. Let α be isolated and $c \in C$. By Lemma 1 there exists a unique arrow $\sigma_c^\alpha : L_\alpha c \longrightarrow Sc$ of A with $\sigma_c^\alpha \lambda_c^{\alpha-1,\alpha} = \sigma_c^{\alpha-1}$. Take any $f : c \longrightarrow c'$ in C and consider the diagram

$$\begin{array}{ccccc}
L_{\alpha-1}c' & \xrightarrow{\lambda_{c'}^{\alpha-1,\alpha}} & L_\alpha c' & \xrightarrow{\sigma_{c'}^{\alpha}} & Sc' \\
\uparrow {\scriptstyle L_{\alpha-1}(f)} & & \uparrow {\scriptstyle L_\alpha(f)} & & \uparrow {\scriptstyle S(f)} \\
L_{\alpha-1}c & \xrightarrow[\lambda_c^{\alpha-1,\alpha}]{} & L_\alpha c & \xrightarrow[\sigma_c^{\alpha}]{} & Sc
\end{array}$$

The left hand square and the outer rectangle commute and therefore $S(f)\sigma_c^\alpha \lambda_c^{\alpha-1,\alpha} = \sigma_{c'}^\alpha L_\alpha(f)\lambda_c^{\alpha-1,\alpha}$. Since this composed arrow factors uniquely through $\lambda_c^{\alpha-1,\alpha}$, $S(f)\sigma_c^\alpha =$ $= \sigma_{c'}^\alpha L_\alpha(f)$ and σ^α is natural. By Lemma 2 σ^α is unique.

Let α be limit and $\beta < \alpha$. Since σ^β is unique, it must hold $\sigma^\beta = \sigma^{\beta+1}\lambda^{\beta,\beta+1}$. Hence for any $c \in$ $\in C$ there exists a unique arrow $\sigma_c^\alpha : L_\alpha c \longrightarrow Sc$ with

$\sigma_c^{\alpha} \lambda_c^{\beta,\alpha} = \sigma_c^{\beta}$. The naturality of σ^{α} can be proved similarly as in the previous case.

Put $\sigma_c^* = \sigma_c^{\gamma(c)}$. Evidently $\sigma^*: L_* \dot{\longrightarrow} S$ is a natural transformation which is the only arrow from L_* to S in $\mathcal{C}_K(T)$.

It remains to show that L_* is left $\mathcal{M}$ -full and left $\mathcal{M}$ -faithful. Let $m \in \mathcal{M}$, $c \in C$ and $\kappa: L_* m \longrightarrow L_* c$. Since $L_* m = Sm$, $\sigma_c^* \kappa: Sm \longrightarrow Sc$ and S is left $\mathcal{M}$ -full, there exists $\kappa': m \longrightarrow c$ with $S(\kappa') = \sigma_c^* \kappa$. It holds $\sigma_c^* \kappa = S(\kappa') = \sigma_c^* L_*(\kappa')$. By Lemma 1 $L_*(h)\kappa = L_*(h) L_*(\kappa')$ for every arrow $h: c \longrightarrow k$ and every $k \in \mathcal{M}$. Since $L_* c$ is a colimit of the diagram $D_{L_*, c}$, it holds $\kappa = L_*(\kappa')$. Thus L_* is left $\mathcal{M}$ -full. The proof of the fact that L_* is left $\mathcal{M}$ -faithful is the same as the first part of the proof of Proposition 1.

The assumption that $\mathcal{M}$ is small and $\mathcal{A}$ cocomplete can be replaced by the supposition that all used colimits exist in $\mathcal{A}$. The supposition that $\mathcal{M}$ generates C is necessary as follows from the following example.

Let C be a full subcategory of the category of ordered sets and isotone maps consisting of a one-point set m and a two-element chain c , $\mathcal{M}$ of a one-point set m . Let $\mathcal{A}$ be a category of upper semilattices and homomorphisms. Let Tm be a one-element upper semilattice. Denote $m = (\{x\}, \leq)$, $c = (\{y, x\}, \leq)$, where $y \leq x$. Let $\underline{y}: m \longrightarrow c$ be the constant arrow with the value y , analogous $\underline{x}$. Let $a = (\{t, u\}, \vee)$, $t \vee u = u$ be a two-element upper semilattice and $\underline{t}, \underline{u} : Tm \longrightarrow a$ as before. Put $Sc = a$, $S(\underline{y}) = \underline{t}$, $S(\underline{x}) = \underline{u}$ and $S'c = a$, $S'(\underline{y}) = \underline{u}$, $S'(\underline{x}) = \underline{t}$. These equalities determine $S, S' \in \mathcal{L}_K(T)$ and an arbitrary element of $\mathcal{L}_K(T)$ is naturally isomorphic with one of them. But there is no natural transformation between S and S'. Hence $\mathcal{L}_K(T)$ has not an initial object.

Proposition 3. Let all suppositions of Proposition 2 be fulfilled and in addition $\mathcal{M}$ be dense in C (left adequate in the sense of Isbell). Then L_* is an initial object in $\mathcal{F}_K(T)$.

Proof: Since the density of $\mathcal{M}$ implies that $\mathcal{M}$ generates C, L is proved to be faithful in the same way as in Proposition 1.

Let $\varkappa : L_* c \longrightarrow L_* c'$ be an arrow of A. Let $m \in \mathcal{M}$. We assign to each arrow $f : m \longrightarrow c$ of C a unique arrow $\tau_m(f) : m \longrightarrow c$ with $L_*(\tau_m(f)) = \varkappa L_*(f)$. We shall show that this assignment gives a natural transformation $\tau : C(K-, c) \stackrel{\cdot}{\longrightarrow} C(K-, c')$ of contravariant functors $\mathcal{M} \longrightarrow Ens$ ($C(Km, c)$ is the set of all arrows $m \longrightarrow c$ of C). Let $g : m' \longrightarrow m$ be an arrow of $\mathcal{M}$ and form the following diagram in Ens

$$\begin{array}{ccc} C(Km, c) & \xrightarrow{\tau_m} & C(Km, c') \\ \Big\downarrow {\scriptstyle C(Kg, c)} & & \Big\downarrow {\scriptstyle C(Kg, c')} \\ C(Km', c) & \xrightarrow{\tau_{m'}} & C(Km', c') \end{array}$$

Let $f \in C(Km, c)$. It is $C(Kg, c')\tau_m(f) = \tau_m(f)g$ and $\tau_{m'} C(Kg, c)(f) = \tau_{m'}(fg)$. Since $L_*(\tau_m(f)g) =$ $= L_*(\tau_m(f))L_*(g) = \kappa L_*(f)L_*(g) = \kappa L_*(fg) = L_*(\tau_{m'}(fg))$, we get $\tau_m(f)g = \tau_{m'}(fg)$ and therefore our diagram commutes. Hence τ is natural and the density of M implies the existence of $\kappa' : c \longrightarrow c'$ with $\tau = C(K-, \kappa')$. Therefore $L_*(\kappa')L_*(f) = L_*(\kappa' f) = L_*(\tau_m(f)) = \kappa L_*(f)$ for any $m \in M$ and $f : m \longrightarrow c$. Hence $L_*(\kappa')\lambda^o_c L_o(f) =$ $= \kappa \lambda^o_c L_o(f)$. Since $L_o c$ is a colimit of the functor $TP : (K \downarrow c) \longrightarrow A$ with the components $L_o(f) : TPf \longrightarrow L_o c$ of the limiting cone, one gets that $L_*(\kappa')\lambda^o_c = \kappa \lambda^o_c$. Since λ^o_c is epi, $L_*(\kappa') = \kappa$ and thus L_* is full.

Corollary 1. Let M be small, dense in C and cogenerate C . Let A be cocomplete and co-well-powered. Then the existence of a left M -full and left M -faithful functor $C \longrightarrow A$ implies the existence of a full and faithful one.

Corollary 2. Let all suppositions of Proposition 3 be fulfilled, T be a full embedding and in addition for every $a \in A$ there exist a proper class of objects of A isomorphic with a . Then L_* is an initial object in $\mathcal{E}_K(T)$.

Proof: Since L_* is full and faithful, $L_* c = L_* c'$ implies that c is isomorphic with c' . Since for every object a of A there is a proper class of objects isomorphic with a , the colimits in the construction of L_* can be chosen such that $L_* c \neq L_* c'$ for isomorphic $c \neq c'$.

A concrete category $(C, \square)$ is a pair consisting of a category C and a faithful functor $\square : C \longrightarrow \mathcal{E}ns$. If $(C, \square)$ is a concrete category, we shall denote the restriction of $\square$ on M again by $\square$. We say that M inductively generates C if for any $c, d \in C$ and any arrow $f : \square c \longrightarrow \square d$ of $\mathcal{E}ns$ $f = \square(f_1)$, for an arrow $f_1 : c \longrightarrow d$ of C if and only if for any $m \in M$ and any arrow $h : m \longrightarrow c$ of C there exists an arrow $h' : m \longrightarrow d$ of C with $\square(h') = f \square(h)$ (see [2]). We say that a concrete category $(C, \square)$ has constants if for any $c, c' \in C$ and any constant function $f : \square c \longrightarrow \square c'$ there exists an arrow $f' : c \longrightarrow c'$ with $\square(f') = f$. If $x \in \square c'$ and $f : \square c \longrightarrow \square c'$ is a constant function with $fy = x$ for any $y \in \square c$, we shall denote this f' by $\underline{x}$.

<u>Lemma 3.</u> Let $(C, \square)$ be a concrete category having constants. Then M is dense if and only if it inductively generates C.

<u>Proof</u>: Let M be dense. Let $c, d \in C$ and $f : \square c \longrightarrow \square d$. Let for any $m \in M$ and any $h : m \longrightarrow c$ there exist an arrow $h' : m \longrightarrow d$ of C with $\square(h') = f \square(h)$. If we put $\tau_m(h) = h'$ for any $h : m \longrightarrow c$, $m \in M$, we obtain a natural transformation $\tau : C(K-, c) \stackrel{\cdot}{\longrightarrow} C(K-, d)$. Hence there exists an arrow $f_1 : c \longrightarrow d$ of C such that $h' = f_1 h$. Let $x \in \square c$. Choosing $h = \underline{x}$, we get $\square(f_1)\square(\underline{x}) = \square(f_1 \underline{x}) = f \square(\underline{x})$. Hence $\square(f_1) = f$.

Let $\mathcal{M}$ inductively generate C. Since C has constants, $\mathcal{M}$ generates C and therefore the functor $C \longrightarrow$ $\longrightarrow \mathcal{E}ns^{\mathcal{M}^{op}}$ given by $c \longmapsto C(K-,c)$ is faithful. It remains to show that it is full. Let $\tau : C(K-,c) \stackrel{\cdot}{\longrightarrow} C(K-,d)$ be a natural transformation. Let $m \in \mathcal{M}$, $x \in \square c$ and consider $\underline{x} : m \longrightarrow c$. Since $\tau_m(\underline{x})g = \tau_m(\underline{x}g) = \tau_m(\underline{x})$ for any $g : m \longrightarrow m$ in $\mathcal{M}$, $\tau_m(\underline{x}) = \underline{x}'$ for some $\underline{x}' : m \longrightarrow d$. Define $f : \square c \longrightarrow \square d$ by $fx = x'$. It can be analogously deduced from the naturality of τ that f does not depend on the choice of m.

We are going to show that $f = \square(f_1)$ for an arrow $f_1 : c \longrightarrow d$ of C. Again, the naturality of τ implies that $\square(\tau_m(h))(x) = (f\,\square(h))(x)$ for any $m \in \mathcal{M}$, $h : m \longrightarrow c$ and $x \in \square m$. Hence $\square(\tau_m(h)) = f\,\square(h)$ and thus $f = \square(f_1)$ for some $f_1 : c \longrightarrow d$ because $\mathcal{M}$ inductively generates C. We have $\square(\tau_m(h)) = \square(f_1)\,\square(h)$ and therefore $\tau_m(h) = f_1 h$. Hence $\tau = C(K-,f_1)$ and the proof is accomplished.

Let $(\mathcal{M}, \square)$ and $(A, \square')$ be concrete categories. A full embedding $T : \mathcal{M} \longrightarrow A$ is called a realization if $\square = \square' T$ (see [5]).

<u>Proposition 4</u>. Let $(C, \square)$, $(A, \square')$ be concrete categories, $(C, \square)$ have constants, $\mathcal{M}$ inductively generate C and T be a realization. Let for any constant $\underline{x} : Lc \longrightarrow Lc'$ of A there exist an $f : c \longrightarrow c'$ such that $L(f) = \underline{x}$. Let a pointwise left Kan extension $L = Lan_K T$ exist and $\mathcal{L}_K(T) \neq \emptyset$. Then L is an initial object

in $\mathcal{F}_K(T)$.

<u>Proof:</u> There exists $S \in \mathcal{L}_K(T)$ and a unique natural transformation $\sigma : L \dot{\longrightarrow} S$ with σK the identity. Let $m \in \mathcal{M}$, $c \in C$ and $g : Lm \longrightarrow Lc$ an arrow of A. There exists $f : m \longrightarrow c$ with $S(f) = \sigma_c g = \sigma_c L(f)$. Let $x \in \square' Lm$. Since $(C, \square)$ has constants and $LK = T$ is a realization, $\underline{x} : Lm \longrightarrow Lm$ is an arrow of A and $g\underline{x}, L(f)\underline{x} : Lm \longrightarrow Lc$ are constants. Thus there exist $h_1, h_2 : m \longrightarrow c$ with $g\underline{x} = L(h_1), L(f)\underline{x} = L(h_2)$. It holds $S(h_1) = \sigma_c L(h_1) = \sigma_c g\underline{x} = \sigma_c L(f)\underline{x} = \sigma_c L(h_2) = S(h_2)$ and therefore $h_1 = h_2$. Hence $g\underline{x} = L(f)\underline{x}$, i.e. $\square'(g)(x) = \square' L(f)(x)$. Therefore $g = L(f)$ and L is left $\mathcal{M}$-full. $\mathcal{M}$ is dense in C by Lemma 3. L is proved to be full in the same way as L_* in the proof of Proposition 3. Since C has aonstants, $\mathcal{M}$ generates C and L is faithful by Proposition 1.

<u>Proposition 5</u>. Let $\mathcal{M}$ be dense in C and $\mathcal{C}_K(T)$ colimit preserving. Then F is a pointwise left Kan extension of T along K .

Proof is evident because $\mathcal{M}$ is dense in C if and only if Id_C together with the identity natural transformation $Id_K : K \longrightarrow K$ is the pointwise left Kan extension of K along K (see [4]).

<u>Proposition 6</u>. Let $\mathcal{M}$ be dense in C and $T : \mathcal{M} \longrightarrow A$ a full embedding. Let $K' : T\mathcal{M} \longrightarrow A$ be the inclusion functor and $T^{-1} : T\mathcal{M} \longrightarrow \mathcal{M}$ the two-sided inverse functor to $T : \mathcal{M} \longrightarrow T\mathcal{M}$. Let the pointwise left Kan extensions

$L = Lan_K\, T$, Id_K and $L' = Lan_{K'}\, T^{-1}$, $Id_{K'}$ exist.
Let L' be left TM -faithful and left TM -full. Then L' is a right adjoint for L .

Proof: It is sufficient to find natural transformations $\eta : Id_C \xrightarrow{\cdot} L'L$, $\varepsilon : LL' \xrightarrow{\cdot} Id_A$ such that the following composites are the identities (of L' resp. L)

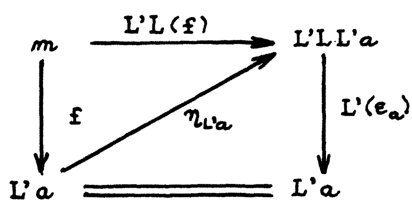
$$L' \xrightarrow{\eta L'} L'LL' \xrightarrow{L'\varepsilon} L \ , \quad L \xrightarrow{L\eta} LL'L \xrightarrow{\varepsilon L} L \ .$$

Let $m \in M$. Putting $\tau_m(f) = L'L(f)$ for each $f : m \longrightarrow c$ we obtain a natural transformation $\tau : C(K-,c) \xrightarrow{\cdot} C(K-,L'Lc)$. Since M is dense, there exists a unique $\eta_c : c \longrightarrow L'Lc$ with $\tau_m(f) = \eta_c f$. Clearly $\eta : Id_C \xrightarrow{\cdot} L'L$ is a natural transformation.

Let $m \in M$, $a \in A$ and $f : m \longrightarrow L'a$ be an arrow of C . Since L' is left TM -full, there exists an arrow $\lambda_f : Tm \longrightarrow a$ of A such that $L'(\lambda_f) = f$. We shall show that $\lambda : TP \xrightarrow{\cdot} a$ is a natural transformation from $(K \downarrow L'a) \xrightarrow{P} M \xrightarrow{T} A$ to the constant functor a . Let h be an arrow of $(K \downarrow L'a)$ with the domain $f : m \longrightarrow L'a$ and the codomain $g : m' \longrightarrow L'a$, i.e. $f = gh$. Then $L'(\lambda_f) = f = gh = L'(\lambda_g)h = L'(\lambda_g)L'T(h) = L'(\lambda_g T(h))$.
Since L' is left TM -faithful, $\lambda_f = \lambda_g T(h)$ and it proves the requested naturality of λ . Since $LL'a$ is a colimit of TP with the components $L(f)$ of the limiting cone, one gets a unique $\varepsilon_a : LL'a \longrightarrow a$ such that $\lambda_f = \varepsilon_a L(f)$ for any $f : m \longrightarrow L'a$ and $m \in M$ .It can be easily shown that $\varepsilon : LL' \xrightarrow{\cdot} Id_A$ is a natural trans-

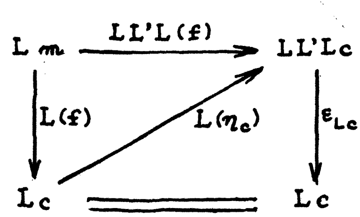
formation. Indeed, $\varepsilon_{a'} LL'(\kappa) = \kappa\varepsilon_a$ for any arrow $\kappa : a \longrightarrow a'$ of A because $\kappa\lambda_f$ are the components of a natural transformation from TP to the constant functor a' and $\varepsilon_{a'} LL'(\kappa)L(f) = \varepsilon_{a'} L(L'(\kappa)f) = \lambda_{L'(\kappa)f} = \kappa\lambda_f$ because $L'(\lambda_{L'(\kappa)f}) = L'(\kappa)f = L'(\kappa)L'(\lambda_f) = L'(\kappa\lambda_f)$.

Consider the following diagram:



The top triangle commutes by the definition of $\eta_{L'a}$. Further $L'(\varepsilon_a)L'L(f) = L'(\varepsilon_a L(f)) = L'(\lambda_f) = f$. Hence $L'(\varepsilon_a)\eta_{L'a} f = f$ and $L'(\varepsilon_a)\eta_{L'a} = 1_{L'a}$ because $L'a$ is a colimit of $T^{-1}P' : (K' \downarrow L'a) \longrightarrow C$. We have proved that $L'\varepsilon \cdot \eta L$ is the identity.

Finally, let $f : m \longrightarrow c$ and take the diagram



The top triangle commutes by the definition of η_c . Further, $\varepsilon_{Lc} LL'L(f) = \lambda_{L'L(f)}$ and $L'L(f) = L'(\lambda_{L'L(f)})$. Hence $\lambda_{L'L(f)} = L(f)$ and in the same way as before we obtain $\varepsilon_{Lc} L(\eta_c) = 1_{Lc}$.

<u>Proposition 7</u>. Let TM be dense in A. Then any two full embeddings from $\mathcal{E}_K(T)$ are naturally isomorphic.

<u>Proof</u>: Let $S, S' \in \mathcal{E}_K(T)$ and $c \in C$. Denote by $K': TM \longrightarrow A$ the inclusion functor. The categories $(K' \downarrow Sc)$ and $(K' \downarrow S'c)$ are isomorphic and therefore the density of TM implies that Sc and $S'c$ are isomorphic. In this way we obtain the natural isomorphism between S and S'.

If $(C, \square)$ and $(A, \square')$ are concrete categories, we can consider the full subcategories of $\mathcal{E}_K(T)$ consisting of all functors F commuting with the forgetful functors $(\square = \square' F)$ or of all realizations. Here density can be replaced by inductive generation and this situation is actually treated in [2].

<u>Applications</u>

A) Let $\mathcal{A}$ be the category of closure spaces (see [1]) and continuous maps. Let $\mathcal{S}^-$ be a category, objects of which are the pairs $a = (\square' a, \mathcal{U})$ where $\square' a$ is a set and $\mathcal{U} \subseteq exp\ \square' a$ and arrows $f: (\square' a, \mathcal{U}) \longrightarrow (\square' b, \mathcal{B})$ correspond with maps $\square'(f): \square' a \longrightarrow \square' b$ such that for $X \in \mathcal{B}$ we have $(\square'(f))^{-1}(X) \in \mathcal{U}$. Let $\square : \mathcal{A} \longrightarrow Ens$, $\square' : \mathcal{S}^- \longrightarrow Ens$ be the forgetful functors. Let u, v be two closure spaces with the same underlying set $\square u = \square v$. We say that $u \leq v$ if there is an arrow $f: v \longrightarrow u$ of $\mathcal{A}$ with $\square(f) = id_{\square u}$. Dual atoms of the lattice of all closure spaces with the same underlying set are called ultraspaces. Any ultraspace is a topological space. Let $\mathcal{U}$

be the full subcategory of $\mathcal{A}$ consisting of all ultraspaces. Realizations of subcategories of $\mathcal{A}$ in $\mathcal{S}^-$ are investigated in [3].

Let C be a full subcategory of $\mathcal{A}$ such that $u \in C$, $w \in \mathcal{U}$, $u \leq w$ implies that $w \in C$. Let $M = C \cap \mathcal{U}$ and $A = \mathcal{S}^-$. Then M inductively generates C. It follows from the fact that whenever a point $x \in \square u$ belongs to the u-closure of some subset $y \subseteq \square u$, then we can find an ultraspace $w \geq u$ such that x belongs to the w-closure of y.

Let $T : M \longrightarrow A$ be a realization. We are going to show that a pointwise left Kan extension $L = Lan_K T$ exists and $Lu = (\square u, \bigcap_{u \leq w \in M} \mathcal{C}l_w)$, where $Tw = (\square w, \mathcal{C}l_w)$. If $u \in C$, $w \in M$ and $f : w \longrightarrow u$ is an arrow of C, we can find a $w_1 \in M$, $w_1 \geq u$ such that there exists an arrow $f_1 : w \longrightarrow w_1$ with $\square(f) = \square(f_1)$. Therefore for any $w \in M$ and any arrow $f : w \longrightarrow u$ of C there is an arrow $\lambda_f : Tw \longrightarrow Lu$ with $\square(f) = \square'(\lambda_f)$. Evidently λ is a cone from the base $(K \downarrow u) \xrightarrow{P} M \xrightarrow{T} A$ to the vertex Lu. Let μ be a cone from TP to $a \in A$. Then $\mu_{\underline{x}}$ is a constant for any constant $\underline{x} : w \longrightarrow u$. Define $h : \square' Lu \longrightarrow \square' a$ by $\underline{hx} = \mu_{\underline{x}}$. There is an arrow $h' : Lu \longrightarrow a$ of A with $\square'(h') = h$ because $\square'(\mu_f) = h$ for any $f : w \longrightarrow u$ with $\square(f) = id_{\square u}$. Hence λ is a limiting cone.

These results can help us in the study of realizations of full subcategories of $\mathcal{A}$ in $\mathcal{S}^-$. Take for instance the full subcategory C of $\mathcal{A}$ consisting of all regular closure T_1-spaces and a realization $T : M \longrightarrow A$. Since C

contains the category of all completely regular topological T_1 -spaces as a full subcategory, it follows from [6] that $Tw = (\square w, \mathcal{O}(w))$, where $\mathcal{O}(w)$ is the system of all open sets of w, for any ultraspace w with a non-measurable underlying set or $Tw = (\square w, \mathcal{L}(w))$, where $\mathcal{L}(w)$ is the system of all closed sets of w, for any such ultraspace. Let the first case occur. Then $Lan_K T(u) = (\square u, \mathcal{O}(u))$ for any u with a non-measurable $\square u$. Hence $Lan_K T$ is not full. By Proposition 4 or by the results of [2] we get the following theorem.

There exists no realization of the category of all regular closure T_1-spaces in $\mathcal{S}^-$.

B) Let C be the category of all Hausdorff topological spaces and continuous maps, M the full subcategory of all regular Hausdorff spaces and $A = \mathcal{S}^-$. Let $\square : C \longrightarrow Ens$ and $\square' : A \longrightarrow Ens$ be the forgetful functors. It is shown in [6] that for any realization $S : C \longrightarrow A$ $Sm = (\square m, \mathcal{O}(m))$ for any $m \in M$ or $Sm = (\square m, \mathcal{L}(m))$ for any $m \in M$. Let $T : M \longrightarrow A$ be a realization such that $Tm = (\square m, \mathcal{L}(m))$ for any $m \in M$. Then $Lan_K Tc = (\square c, \mathcal{L}(c))$ for any $c \in C$. Hence $Lan_K T$ is a realization and it is an initial object in $\mathcal{E}_K(T)$.

M is a reflexive subcategory of C. Denote by $F : C \longrightarrow M$ a left adjoint to the inclusion functor $K : M \longrightarrow C$ and $\eta : Id_C \xrightarrow{\cdot} KF$ the unit of this adjunction. Then a pointwise right Kan extension exists and is equal to TF. The full subcategory of $\mathcal{E}_K(T)$ consisting of all funcctors commuting with the forgetful functors has a terminal object R which is defined as follows: $Rc = (\square c, \mathcal{R}(c))$, where $\mathcal{R}(c) = \{\eta_c^{-1} x / x \in \mathcal{L}(Fc)\}$. The functor R is right $\mathcal{M}$-full and right $\mathcal{M}$-faithful and thus is a terminal object in the full subcategory of $\mathcal{C}_K(T)$ consisting of all such functors. By [6] R is not a full embedding because $\mathcal{R}(c)$ is not a subbasis for $\mathcal{L}(c)$. The problem of the existence of a terminal object in $\mathcal{C}_K(T)$ is in close connection with the open problem concerning the number of realizations of C in A.

C) Let C be the category of ordered sets and isotone maps, $\mathcal{M}$ a full subcategory of C having a single object, namely a two-element chain and A the category of semigroups and homomorphisms. Let $\square : C \longrightarrow \mathcal{E}ns$ and $\square' : A \longrightarrow \mathcal{E}ns$ be the forgetful functors. $\mathcal{M}$ is dense in C and cogenerates C. Let T assign to the two-element chain a two-element upper semilattice. Clearly $T : \mathcal{M} \longrightarrow A$ is a realization. M. Sekanina has constructed in [7] a full embedding $H : C \longrightarrow A$ extending T as follows: Hc is the free semigroup with the generating set $\square c$ and with relations $x \cdot y = x = y \cdot x \Longleftrightarrow x, y \in \square c$, $x \geq y$. It can be easily shown that H is a pointwise left Kan extension of T along K. Therefore H is an initial object in $\mathcal{C}_K(T)$. Let K' and T^{-1} be as in Proposition 6. $Lan_{K'} T^{-1} = L'$ assigns to each $a \in A$ the set Ia of all idempotents of a with the following ordering: $x, y \in Ia$, $x \geq y \Longleftrightarrow x \cdot y = x = y \cdot x$. This ordering is considered in the theory of semigroups. For instance, if $\mathcal{a}$ is an upper semilattice, then $L'\mathcal{a}$ is its ordered set. Clearly L' is left TM -full and left TM -faithful. By Proposition 6 L' is a right adjoint for H . By Proposition 5 H is up to the natural isomorphism the only colimit preserving full embedding from $\mathcal{C}_K(T)$. There is no limit preserving full embedding $C \longrightarrow A$ inducing a realization on M because a semigroup product of two semilattices is an idempotent semigroup.

D) A similar situation is in the following case (see [8]). C is the category of graphs and arrows are mappings preserving the relation "between", M is a full subcategory of all trees and A is the category of ternary algebras and homomorphisms. M is dense in C and cogenerates C . In [8] it is constructed a realization $T: M \longrightarrow A$ and $Lan_K T$ is proved to be a full embedding. But in this case it has not a right adjoint.

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Přírodovědec.fakulta
university J.E. Purkyně
66295 Brno, Janáčkovo nám.2 a
Československo

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