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ON SKEW LATTICES II

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Abstract: In the present paper prelattices are studied and a method is given which enables to transfer some results of lattice theory on theorems about prelattices. As an application of this method some results concerning distributive and modular prelattices are given.

Key words: Skew lattice, primitive class.

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1. Preliminaries. An algebra $\mathfrak{N} = \langle N, \wedge, \vee \rangle$ is called a nest iff for all $a, b \in N$ $a \wedge b = a$ and $a \vee b = b$. A skew lattice $\mathfrak{M}$ is said to be a prelattice iff for all $a, b, c \in M$

$$(c \vee a \vee b) \wedge (b \vee a) = a \vee b ,$$

$$(a \wedge b) \vee (b \wedge a \wedge c) = c \wedge a .$$

Evidently any lattice is a prelattice and any nest is a prelattice. M.D. Gerhardts proved ([3]) that for a skew lattice to be a prelattice it is necessary and sufficient to be isomorphic to the direct product of a lattice and a nest. It is also known ([3]) that the relation $\sim$ defined by

$$a \sim b \quad \text{iff} \quad a \wedge b = b \wedge a$$

is a congruence relation on any prelattice. If a, b are elements of a prelattice then $a \sim b$ is equivalent to $a \vee b = b \vee a$. One can easily show that if $\mathcal{M}$ is a prelattice then $\mathcal{M}$ is isomorphic to $\mathcal{M}/\equiv \times \mathcal{M}/\sim$ ($\mathcal{M}/\equiv$ is a lattice and $\mathcal{M}/\sim$ is a nest).

All the notations not defined below can be found in the paper [4].

2. Prelattices.

2.1. Proposition. The following two conditions are equivalent for a skew lattice $\mathcal{M}$:

(1) Every equivalence relation on M is a congruence relation on $\mathcal{M}$.

(2) $\mathcal{M}$ is a nest or the two-element lattice.

Proof. It can be easily verified that (2) implies (1). Let $\mathcal{M}$ be a skew lattice of cardinality at least 3 and suppose that (1) holds. We shall show that $a \leq b$ for all $a, b \in M$. The relation $\theta_{x,y} = id_M \cup (x,y) \cup (y,x)$ is an equivalence relation on M and thus $\theta_{x,y}$ is a congruence relation on $\mathcal{M}$. Assume that there exist a , $b \in M$ such that $a \neq a \wedge b$. Since $(a \wedge b, b \wedge b) \in \theta_{a,b}$ and $(a \vee b, b \vee b) \in \theta_{a,b}$ we have $a \wedge b = b$ and $a \vee b = a$. We can easily see that $b \wedge a = b \wedge (a \wedge b) = b$ and $b \vee a = a \vee (b \vee a) = a$. Let c be an element of M different from a and b . Since $(b \wedge a, b \wedge c) \in \theta_{a,c}$, $(a \vee b, a \vee c) \in \theta_{b,c}$ and $b \wedge a = b$, $a \vee b = a$, we get $b \wedge c = b$ and $a \vee c = a$. It follows from $(a, b) \in \theta_{a,b}$ that

$(a \vee c, b \vee c) \in \theta_{a,b}$, i.e. $(a, c) \in \theta_{a,b}$; this contradiction completes the proof.

2.2. Corollary. A nest is subdirectly irreducible if and only if it has at most two elements.

2.3. Theorem. A skew lattice $\mathcal{M}$ is a prelattice if and only if the relation $\sim$ is a congruence relation on $\mathcal{M}$.

Proof. It suffices to prove that if $\sim$ is a congruence relation on $\mathcal{M}$ then $\mathcal{M}$ is a prelattice. Evidently, if $\sim$ is a congruence relation on $\mathcal{M}$ then $\mathcal{M}/\sim$ is a nest. Let us denote the natural homomorphism of $\mathcal{M}$ onto $\mathcal{M}/\equiv$ and that of $\mathcal{M}$ onto $\mathcal{M}/\sim$ by ν and μ , respectively. Define $\varphi : M \longrightarrow M/\equiv \times M/\sim$ by $m\varphi = (m\nu, m\mu)$. The mapping φ is injective. Indeed, if $a, b \in M$ are such that $a\varphi = b\varphi$ then $a \equiv b$ and $a \sim b$; thus $a = a \wedge b = b \wedge a = b$. Since φ is clearly a homomorphism of $\mathcal{M}$ into $\mathcal{M}/\equiv \times \times \mathcal{M}/\sim$, we get that $\mathcal{M}$ can be embedded into the prelattice $\mathcal{M}/\equiv \times \mathcal{M}/\sim$. Thus $\mathcal{M}$ is a prelattice. One can easily verify that φ is an isomorphism of $\mathcal{M}$ onto $\mathcal{M}/\equiv \times \mathcal{M}/\sim$.

Given a class of lattices $\mathcal{K}$, denote by $J(\mathcal{K})$ the class of all prelattices $\mathcal{M}$ such that $\mathcal{M}/\equiv \in \mathcal{K}$. It is easy to show that $J(\mathcal{K})$ is the intersection of the class $\mathcal{S}(\mathcal{K})$ and of the clas of all prelatiices. A skew lattice belongs to $J(\mathcal{K})$ if and only if it is isomorphic to the direct product of a lattice from $\mathcal{K}$ and of a nest. It is evident that if $\mathcal{K}$ contains the one-element lattice then all nests belong to $\mathcal{J}(\mathcal{K})$.

Since every subdirectly irreducible prelattice has to be a lattice or a nest, we have

2.4 <u>Theorem</u>. Let $\mathcal{K}$ be a class of lattices containing the one-element lattice. Then the subdirectly irreducible skew lattices from $\mathcal{J}(\mathcal{K})$ are exactly the subdirectly irreducible lattices from $\mathcal{K}$ and the two-element nest.

Let $\mathcal{L}_L$, $\mathcal{L}_{SL}$ and $\mathcal{L}_{PL}$ denote the lattice of all primitive classes of lattices, that of skew lattices and that of prelattices, respectively.

Since primitive classes of algebras are uniquely determined by their subdirectly irreducible algebras, we get

2.5. <u>Theorem</u>. Let $\mathcal{K}$ be a primitive class of lattices. Then $\mathcal{K}$ is covered by $\mathcal{J}(\mathcal{K})$ in $\mathcal{L}_{SL}$.

2.6. <u>Theorem</u>. The lattice $\mathcal{L}_{PL}$ is isomorphic to the direct product of $\mathcal{L}_L$ and of the two-element lattice $\mathbf{2}$.

<u>Proof</u>. Let $\mathbf{2} = \langle \{0,1\} ; 0 \leq 1 \rangle$ and define $\varphi : L_L \times \{0,1\} \longrightarrow L_{PL}$ by

$$(\mathcal{K},0)\varphi = \mathcal{K} \ , \ (\mathcal{K},1)\varphi = \mathcal{J}(\mathcal{K}) \ .$$

Clearly φ is an isomorphism of $\mathcal{L}_L \times \mathbf{2}$ onto $\mathcal{L}_{PL}$.

3. <u>Main results</u>. The theory of nests and that of prelattices will ne denoted by T_N and T_{PL} , respectively. A formula φ is said to be a consequence of a

theory T iff φ is satisfied in every model of T. The set of all consequences of a theory T is denoted by $Cn(T)$. If $\mathfrak{M}$, $\mathfrak{N}$ are skew lattices then the natural homomorphism of $\mathfrak{M} \times \mathfrak{N}$ onto $\mathfrak{M}$ and that of $\mathfrak{M} \times \mathfrak{N}$ onto $\mathfrak{N}$ will be denoted by $_1$ and $_2$, respectively.

The following theorem follows immediately from Theorem 3.8 and Theorem 3.10 of 4 .

3.1. Theorem. Let K be an axiomatic (elementary) class of lattices. Then the class $J(K)$ is also axiomatic (elementary). Moreover, if $K = Mod(T_L \cup T)$ where T is a theory then $J(K) = Mod(T_{PL} \cup T^*)$. If K is a variety (quasi-variety) of lattices then $J(K)$ is a variety (quasi-variety) of prelattices.

3.2. Theorem. Let T_1, T_2 be theories. We have two equivalent statements:

(1) $Mod(T_L \cup T_1) \subseteq Mod(T_L \cup T_2)$;

(2) $Mod(T_{PL} \cup T_1^*) \subseteq Mod(T_{PL} \cup T_2^*)$.

Proof. By 3.9 of [4].

A formula φ is called a J-formula if the following condition (J) holds:
Whenever $\mathcal{L}$ is a lattice and whenever $\mathfrak{N}$ is a nest and α is a mapping of X into $L \times N$, the formula φ is satisfied by α in $\mathcal{L} \times \mathfrak{N}$ if and only if φ is satisfied by α_1 in $\mathcal{L}$.

By the J-theory we meant a theory containing only J-formulas.

3.3. <u>Lemma</u>. A J-formula which is satisfied in the one-element lattice is a consequence of T_N

<u>Proof</u>. A J-formula which is satisfied in the one-element lattice is satisfied by every α in the direct product of one-element lattice and of a nest, i.e. it is satisfied in every nest.

3.4. <u>Lemma</u>. Let φ be a formula such that the following condition (H) holds:
Whenever $\mathfrak{M}_1, \mathfrak{M}_2$ are skew lattices and α is a mapping of X into $M_1 \times M_2$ then φ is satisfied by α in $\mathfrak{M}_1 \times \mathfrak{M}_2$ if and only if φ is satisfied by α_1 in $\mathfrak{M}_1$ and φ is satisfied by α_2 in $\mathfrak{M}_2$.
Let φ be a consequence of T_N. Then φ is a J-formula.

<u>Proof</u>. It is easy to show that φ satisfies the condition (J).

Since equations and quasi-equations satisfy (H) and since they are satisfied in the one-element lattice, we have

3.5. <u>Proposition</u>. Let φ be an equation or a quasi-equation. Then φ is a J-formula if and only if $\varphi \in Cn(T_N)$.

3.6. <u>Proposition</u>. Let φ_1, φ_2 be J-formulas. Then $\neg\varphi_1$, $\varphi_1 \& \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi_1 \longrightarrow \varphi_2$ are J-formulas.

<u>Proof</u>. One can verify without difficulty that the ne-

gation of a J-formula is also a J-formula. We shall now show that the conjunction of two J-formulas is a J-formula. Let φ_1, φ_2 be J-formulas. Suppose that a lattice $\mathcal{L}$, a nest $\mathcal{N}$ and $\alpha : X \longrightarrow L \times N$ are given. The formula $\varphi_1 \& \varphi_2$ is satisfied by α in $\mathcal{L} \times \mathcal{N}$ if and only if both formulas φ_1 and φ_2 are satisfied by α in $\mathcal{L} \times \mathcal{N}$. Since φ_1, φ_2 are J-formulas, they are satisfied by α in $\mathcal{L} \times \mathcal{N}$ if and only if they are satisfied by α_1 in $\mathcal{L}$. So we have that $\varphi_1 \& \varphi_2$ is satisfied by α in $\mathcal{L} \times \mathcal{N}$ if and only if $\varphi_1 \& \varphi_2$ is satisfied by α_1 in $\mathcal{L}$. Thus the formula satisfies the condition (J).

3.7. Proposition. Let φ be a J-formula and let x be a variable. Then $(\forall x)\varphi$ and $(\exists x)\varphi$ are J-formulas.

Proof. We shall show that $(\exists x)\varphi$ is a J-formula. Suppose that a lattice $\mathcal{L}$, a nest $\mathcal{N}$ and $\alpha : X \longrightarrow L \times N$ are given. If the formula $(\exists x)\varphi$ is satisfied by α in $\mathcal{L} \times \mathcal{N}$, then there exists $\beta : X \longrightarrow L \times N$ such that $\alpha/X \setminus \{x\} = \beta/X \setminus \{x\}$ and φ is satisfied by β in $\mathcal{L} \times \mathcal{N}$. Since φ is a J-formula and $\beta_1/X \setminus \{x\} = \alpha_1/X \setminus \{x\}$ we get that the formula $(\exists x)\varphi$ is satisfied by α_1 in $\mathcal{L}$. Conversely, suppose that $(\exists x)\varphi$ is satisfied by α_1 in $\mathcal{L}$. Then there exists $\beta : X \longrightarrow L$ such that $\alpha_1/X \setminus \{x\} = \beta/X \setminus \{x\}$ and φ is satisfied by β in $\mathcal{L}$. It can be easily shown that there exists

$\gamma : X \longrightarrow L \times N$ such that $\gamma/X \setminus \{x\} = \alpha/X \setminus \{x\}$ and $\gamma \;_1 = \beta$. Since φ is a J-formula, it is satisfied by γ in $\mathcal{L} \times \mathcal{N}$. This completes the proof.

3.8. Theorem. Let $\mathcal{L}$ be a lattice and let $\mathcal{N}$ be a nest. If φ is a J-formula, then φ is satisfied in $\mathcal{L} \times \mathcal{N}$ if and only if φ is satisfied in $\mathcal{L}$.

Proof. It is evident that every mapping of X into L can be represented as the product of a mapping of X into $L \times N$ and of the mapping $\;_1$. Combining this fact with the condition (J) one can prove the theorem 3.8 without further difficulties.

Since every prelattice $\mathcal{M}$ is isomorphic to $\mathcal{M}/\equiv \times \mathcal{M}/\sim$, we have

3.9. Theorem. A prelattice $\mathcal{M}$ is a model of a J-theory T if and only if the lattice $\mathcal{M}/\equiv$ is a model of T.

3.10. Corollary. Let T be a J-theory. Then

$$\mathrm{Mod}\,(T_{PL} \cup T^*) = \mathrm{Mod}\,(T_{PL} \cup T) .$$

If we combine Theorem 3.2 with Corollary 3.10, we get the following result.

3.11. Theorem. Let T_1, T_2 be J-theories. Then the following conditions are equivalent.

(1) $\mathrm{Mod}\,(T_L \cup T_1) \subseteq \mathrm{Mod}\,(T_L \cup T_2)$;

(2) $\mathrm{Mod}\,(T_{PL} \cup T_1) \subseteq \mathrm{Mod}\,(T_{PL} \cup T_2)$.

3.12. <u>Theorem.</u> Let $\mathcal{K}$ be a class of lattices and let $\{\mathcal{L}_i ; i \in I\}$ be a family of lattices. We have two equivalent statements:

(1) If $\mathcal{L} \in \mathcal{K}$, then $\mathcal{L}$ contains no sublattice isomorphic to some $\mathcal{L}_i$ $(i \in I)$.

(2) If $\mathcal{M} \in \mathcal{J}(\mathcal{K})$, then $\mathcal{M}$ contains no subprelattice isomorphic to some $\mathcal{L}_i$ $(i \in I)$.

<u>Proof.</u> Since every lattice is a prelattice it suffices to prove that (1) implies (2). Assume (1) and let $\mathcal{M} \in$ $\in \mathcal{J}(\mathcal{K})$ be such that some $\mathcal{L}_i$ $(i \in I)$ can be embedded into $\mathcal{M}$. It is easy to show that $\mathcal{L}_i$ can be also embedded into $\mathcal{M}/\equiv$. This contradiction completes the proof.

3.13. <u>Theorem.</u> Let $\mathcal{K}$ be a class of lattices and let $\{\mathcal{L}_i ; i \in I\}$ be a family of lattices. The following statements are equivalent:

(1) If a lattice $\mathcal{L}$ has no sublattice isomorphic to some $\mathcal{L}_i$ $(i \in I)$, then $\mathcal{L} \in \mathcal{K}$.

(2) If a prelattice $\mathcal{M}$ has no subprelattice isomorphic to some $\mathcal{L}_i$ $(i \in I)$, then $\mathcal{M} \in \mathcal{J}(\mathcal{K})$.

<u>Proof.</u> It is evident that (2) implies (1). Assume (1) and let $\mathcal{M}$ be a prelattice which has no subprelattice isomorphic to some $\mathcal{L}_i$ $(i \in I)$. Since $\mathcal{M}$ is isomorphic to $\mathcal{M}/\equiv \times \mathcal{M}/\sim$ the lattice $\mathcal{M}/\equiv$ is a subprelattice of $\mathcal{M}$ and so it cannot contain a sublattice isomorphic to some $\mathcal{L}_i$ $(i \in I)$. Thus we have $\mathcal{M}/\equiv \in \mathcal{K}$ and hence $\mathcal{M} \in \mathcal{J}(\mathcal{K})$.

4. Distributive and modular prelattices. Using preceding results we shall obtain in this section some generalizations of certain results concerning distributive and modular lattices.

The class of all distributive lattices and that of all modular lattices will be denoted by K_D and K_M, respectively.

4.1. Definition. A skew lattice $\mathfrak{M}$ is called distributive iff for all $a, b, c \in M$

$$(b \vee c) \wedge a = (b \wedge a) \vee (c \wedge a) ,$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) .$$

One can easily show that the following theorem holds.

4.2. Theorem. The following conditions are equivalent for a skew lattice $\mathfrak{M}$:

(1) $\mathfrak{M}$ is distributive.

(2) $\mathfrak{M}$ is a weak distributive prelattice.

(3) $\mathfrak{M}$ belongs to $\mathfrak{J}(K)$.

4.3. Theorem. For a prelattice to be distributive, each of the following conditions is necessary and sufficient:

(1) $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ imply $b = c$.

(2) $b \wedge a = c \wedge a$ and $b \vee a = c \vee a$ imply $b = c$.

(3) $b \wedge a = c \wedge a$ and $a \vee b = a \vee c$ imply $b = c$.

Proof. It follows from Proposition 3.5 that the following formulas $\varphi_1, \varphi_2, \varphi_3$ are J-formulas:

$$\varphi_1 = ((x_1 \wedge x_2 = x_1 \wedge x_3 \ \& \ x_1 \vee x_2 = x_1 \vee x_3) \longrightarrow x_2 = x_3) \ .$$

$$\varphi_2 = ((x_2 \wedge x_1 = x_3 \wedge x_1 \ \& \ x_2 \vee x_1 = x_3 \vee x_1) \longrightarrow x_2 = x_3) \ .$$

$$\varphi_3 = ((x_2 \wedge x_1 = x_3 \wedge x_1 \ \& \ x_1 \vee x_2 = x_1 \vee x_3) \longrightarrow x_2 = x_3) \ .$$

Since each of these formulas is equivalent to the distributivity in the theory of lattices, each of the formulas $\varphi_1, \varphi_2, \varphi_3$ is equivalent to the distributivity in the theory of prelattices as it follows easily by 3.11.

4.4. <u>Theorem</u>. A skew lattice $\mathcal{M}$ is a distributive lattice if and only if for all $a, b, c \in M$ the condition $(*)$ holds:

$(*)$ $a \wedge b = a \wedge c$ and $b \vee a = c \vee a$ imply $b = c$.

<u>Proof</u>. It suffices to prove that every skew lattice satisfying the condition $(*)$ is commutative. Let $\mathcal{M}$ be a skew lattice satisfying $(*)$ and let $a, b, c \in M$. Since

$$a \wedge (b \wedge a) = a \wedge b = a \wedge (a \wedge b) \ ,$$

$$(b \wedge a) \vee a = a = (a \wedge b) \vee a \ ,$$

$$a \wedge (a \vee b) = a = a \wedge (b \vee a) \ ,$$

$$(a \vee b) \vee a = b \vee a = (b \vee a) \vee a \ ,$$

we have $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.

4.5. <u>Theorem</u>. A skew lattice $\mathcal{M}$ is distributive if and only if for all $a, b, c \in M$

$$(a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c) \ .$$

Proof. It is known that the equation

(i) $(x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3) = (x_1 \vee x_2) \wedge$
$\wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3)$

is equivalent to the distributivity in the theory of lattices. Since the equation (i) is a J-formula, it is equivalent to the distributivity in the theory of prelattices. So it is sufficient to prove that every skew lattice satisfying (i) is a prelattice. Suppose that $\mathcal{M}$ is a skew lattice satisfying (i). It is easy to show that for all $a, b \in M$ $a \vee (a \wedge b) = a$ and $(b \vee a) \wedge a = a$. If a, b, c are elements of M, then

$$(c \vee a \vee b) \wedge (b \vee a) = c \vee (a \vee b)) \wedge (c \vee b \vee a) \wedge ((a \vee b) \vee (b \vee a)) =$$
$$= (c \wedge (a \vee b)) \vee (c \wedge (b \vee a)) \vee ((a \vee b) \wedge (b \vee a)) = a \vee b \ .$$

We can show similarly that

$$(a \wedge b) \vee (b \wedge a \wedge c) = b \wedge a \ .$$

The proof is thus complete.

The following theorem gives a characterization of prelattices by their subprelattices.

4.6. Theorem. A prelattice is distributive if and only if it has no subprelattice isomorphic to the lattice on Fig. 1 or to the lattice on Fig. 2.

Proof. The theorem follows immediately from Theorem 3.12 and Theorem 3.13.

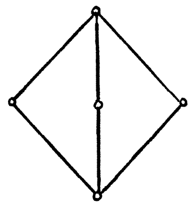

Fig. 1

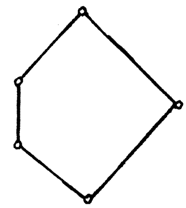

Fig. 2

4.7. <u>Definition</u>. A skew lattice $\mathfrak{M}$ is called modular iff for all $a, b, c \in M$

$$(b \vee (c \wedge a)) \wedge a = (b \wedge a) \vee (c \wedge a) \quad ,$$

$$a \vee ((a \vee c) \wedge b) = (a \vee c) \wedge (a \vee b) \quad .$$

The proofs of the following theorems are similar to the ones of the corresponding theorems about distributive prelattices and so they are omitted.

4.8. <u>Theorem</u>. For a skew lattice $\mathfrak{M}$, the following conditions are equivalent:

(1) $\mathfrak{M}$ is modular.

(2) $\mathfrak{M}$ is weak modular prelattice.

(3) $\mathfrak{M}$ belongs to $J(K)$.

4.9. <u>Theorem</u>. For a prelattice to be modular each of the following conditions is necessary and sufficient:

(1) $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ and $b \leq c$ imply $b = c$.

(2) $b \wedge a = c \wedge a$ and $b \vee a = c \vee a$ and $b \leq c$ imply $b = c$.

(3) $b \wedge a = c \wedge a$ and $a \vee b = a \vee c$ and $b \leq c$ imply $b = c$.

4.10. Theorem. A skew lattice $\mathfrak{M}$ is a modular lattice if and only if for all $a, b, c \in M$ the following condition holds:

$a \wedge b = a \wedge c$ and $b \vee a = c \vee a$ and $b \leq c$ imply $b = c$.

4.11. Theorem. A prelattice is modular if and only if it contains no subprelattice isomorphic to the lattice on Fig.2.

R e f e r e n c e s

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