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ON HADAMARD DIFFERENTIABILITY

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Abstract: In this paper, a necessary and sufficient condition for a Gâteaux differentiable mapping to be Hadamard differentiable is given. It is proved as a consequence that a mapping f with a Gâteaux variation $Vf(x, h)$ which is jointly continuous at a point $(x_0, 0)$ possesses an Hadamard derivative at the point x_0.

Key words: Nonlinear mappings in normed linear spaces, Gateaux differentiability, Hadamard differentiability.

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1. In 1923 Hadamard published a note in which a new method how a differential of a function can be defined, was shown. Later on, his idea was precised and generalized by Fréchet and now we can state the following definition of differentiability in the generalized sense of Hadamard (see [2],[3],[4],[6]):

Definition. Let X, Y be normed linear spaces. An operator $f: X \longrightarrow Y$ is said to have an Hadamard differential at a point $x_0 \in X$ if there exists a continuous linear mapping $L: X \longrightarrow Y$ such that for any continuous mapping $\varphi: [0,1] \longrightarrow X$ for which $\varphi'(0+)$ exists and $\varphi(0) = x_0$, the mapping $F(t) = f(\varphi(t))$ is differentiable at $t = 0+$ and $F'(0+) = L\varphi'(0+)$. The mapping L is called the Hadamard derivative of f at x_0 and $Lg'(0+)$ is called the Hadamard differential of f at x_0. For abbreviation, we shall often write " H-derivative" and "H-differential" only.

The idea of Hadamard differentiability was later transferred to topological spaces (for example, by A.D. Michal, Ky Fan, M. Balanzat, Long de Foglio). Simultaneously, in connection with the development of a differential calculus in topological spaces (by Gil de Lamadrid, Sebastião e Silva, H.H. Keller, M. Sova, the above mentioned authors and others - see [6] for references), a notion of a compact differentiability appeared (see [2],[3],[6],[7], [8]). M. Sova proved in [8] that both the notions of Hadamard differentiability and compact differentiability are equivalent in the case of metrizable linear spaces. We state here this fact as

<u>Theorem</u> (Sova [8]). An operator $f: X \longrightarrow Y$ is H-differentiable at $x_0 \in X$ with an H-derivative $f'(x_0): X \longrightarrow Y$ if and only if $\lim_{t \to \infty} \frac{1}{t} \omega(x_0, th) = 0$ uniformly with respect to $h \in C$ for every closed convex symmetric sequentially compact set $C \subset X$, where $\omega(x_0, h) = f(x_0 + h) - f(x_0) - f'(x_0)h \quad (h \in X)$.

Various properties of H-differentials are listed, for example, in [4] and [6]. We recall here only the facts that Fréchet differentiability (see [6] or [9] for a definition) implies H-differentiability, H-differentiability implies Gâteaux differentiability (see [6] or [9]; we shall further write G-differentiability only), the notions of Fréchet and

Hadamard differentiability are equivalent in the case of finite dimensional spaces and a G-differentiable Lipschitzian mapping is H-differentiable (see [4],[6]).

The last fact and the theorem of Sova quoted above are the only known types of the criteria for Hadamard differentiability. The purpose of this paper is to give some other conditions under which a G-differentiable mapping is H-differentiable.

2. Throughout the paper, let X and Y be normed linear real spaces. However, it is evident that our results are valid (with a few slight formal modifications) in metrizable topological linear spaces, too. By R , the real line is denoted.

<u>Theorem 1</u>. Let x_0 be a point of X, $M \subset X$ a neighbourhood of x_0, $f : M \longrightarrow Y$ and assume that there is a G-derivative $f'(x_0)$ of f at the point x_0. Then f has an H-derivative at x_0 if and only if the functional φ_h

$$\begin{cases} \varphi_h(t,k) = \| \frac{1}{t} [f(x_0+th) - f(x_0+tk)] \| & (t \neq 0) \\ \varphi_h(0,k) = 0 \end{cases}$$

is continuous at the point $(0,h) \in R^+ \times X$ for every fixed $h \in X$, i.e. if and only if for every $\varepsilon > 0$ and $h \in X$, there are $\delta > 0$ and $\eta > 0$ such that

$$\| \frac{1}{t} [f(x_0+th) - f(x_0+tk)] \| < \varepsilon$$

for all $k \in X$ with $\| k - h \| < \eta$ and all $t \in (0,\delta)$.

<u>Proof</u>. We can suppose that M is convex and open. Let

$\Delta_h > 0$ be sufficiently small so that $x_0 + \Delta_h \in M$ and set

$$\tilde{\omega}(h,t) = \frac{1}{t}[f(x_0+ht) - f(x_0)] - f'(x_0)h$$

for $h \in X$ and $t \in (0, \Delta_h)$.

1° Suppose our condition is satisfied and let a number $\varepsilon > 0$ and a closed convex symmetric sequentially compact set C be given. It follows from G-differentiability of f at x_0 that there is $\delta'_h \in (0, \Delta_h)$ for every given $h \in X$ such that

$$\|\tilde{\omega}(h,t)\| \leq \frac{\varepsilon}{3} \tag{1}$$

for all $t \in (0, \delta'_h)$. Moreover, there is $\eta'_h > 0$ such that

$$\|f'(x_0)(h-k)\| \leq \frac{\varepsilon}{3} \tag{2}$$

whenever $\|h-k\| < \eta'_h$ because of continuity of $f'(x_0): X \longrightarrow Y$.

By our assumption, we can choose numbers $\delta_h \in (0, \delta'_h)$ and $\eta_h \in (0, \eta'_h)$ for every $h \in X$ so that $x_0 + t k \in M$ and

$$\left\| \frac{1}{t}[f(x_0+th) - f(x_0+tk)] \right\| \leq \frac{\varepsilon}{3} \tag{3}$$

whenever $\|h-k\| < \eta_h$ and $t \in (0, \delta_h)$. It follows then from (1),(2),(3) that for every given $\varepsilon > 0$ and $h \in X$,

$$(4)\qquad \|\tilde{\omega}(k,t)\| \le \|\tilde{\omega}(h,t)\| + \|\tilde{\omega}(h,t) - \tilde{\omega}(k,t)\| \le$$
$$\le \|\tilde{\omega}(h,t)\| + \|f'(x_0)(h-k)\| +$$
$$+ \|\frac{1}{t}[f(x_0+th) - f(x_0+tk)]\| \le \varepsilon$$

for all $k \in X$ with $\|k - h\| < \eta_h$ and all $t \in (0, \delta_h)$.

Denote $B(h) = \{k \in X : \|k - h\| < \eta_h\}$. These sets are open and $\bigcup_{h \in C} B(h) \supset C$. Since C is compact, we can choose a finite number of elements $h_1, \dots, h_n \in C$ such that $\bigcup_{i=1}^{n} B(h_i) \supset C$; set $\delta_C = \min\{\delta_{h_1}, \dots, \delta_{h_n}\}$. It follows then from (4) that

$$\|\tilde{\omega}(k,t)\| \le \varepsilon$$

for all $t \in (0, \delta_C)$ and all $k \in C$ which means by the theorem of Sova quoted above that $f'(x_0)$ is an H-derivative of f at x_0.

2° On the other hand, suppose f has an H-derivative $f'(x_0)$ at x_0. If the condition of our theorem were not valid, such a number $\varepsilon_0 > 0$, an element $h_0 \in X$ and sequences $\{h_n\} \subset X$ and $\{t_n\} \subset (0, \Delta_h)$ would exist that

$$\|h_n - h_0\| < \frac{1}{n},\ t_n < \frac{1}{n}$$

and

$$(5)\qquad \|\frac{1}{t_n}[f(x_0 + t_n h_0) - f(x_0 + t_n h_n)]\| > \varepsilon_0$$

for all $n = 1, 2, \dots$.

The mapping $f'(x_0): X \longrightarrow Y$ is continuous at h_0 by the definition so that there exists a number n'_0 such that

$$\| f'(x_0) h_n - f'(x_0) h_0 \| \leq \frac{\varepsilon_0}{3} \tag{6}$$

for all $n \geq n'_0$. Since $h_n \longrightarrow h_0$,the set $C = \{ h_m : m = 0,1,2,\ldots\}$ is compact and so by the Mazur's theorem the closed convex hull $\tilde{C}$ of $C \cup (-C)$ is symmetric and compact, too. By the theorem of Sova, $\tilde{\omega}(h,t) \longrightarrow 0$ if $t \rightarrow 0$ uniformly with respect to $h \in \tilde{C}$ and hence we can find n''_0 such that

$$\| \tilde{\omega} (h_m , t_n) \| \leq \frac{\varepsilon_0}{3} \tag{7}$$

for all $n \geq n''_0$ and all $m \geq 0$.

We conclude from (6) and (7) that there is n_0 such that

$$\| \frac{1}{t_n} [f(x_0 + t_n h_0) - f(x_0 + t_n h_n)] \| \leq$$

$$\leq \| \tilde{\omega} (h_0 , t_n) \| + \| \tilde{\omega} (h_n , t_n) \| +$$

$$+ \| f'(x_0) h_n - f'(x_0) h_0 \| \leq \varepsilon_0$$

for all $n \geq n_0$. However, this is a contradiction to (5). The theorem is proved.

<u>Corollary</u>. Let $M \subset X$ be a set with a non-empty interior and let $f: M \longrightarrow Y$ be a mapping having a G-derivative at some interior point x_0 of M . Suppose there is a functional $\varphi : X \longrightarrow R$ such that

$$\| f(x_0 + u) - f(x_0 + v) \| \leq \varphi (u - v)$$

for all $u, v \in M$ and let $\lim_{h \to \infty} \varphi(h) = 0$ and

$\varphi(th) \leq \alpha(t)\, \varphi(h)$ for all $h \in M$ and $t \in (0, \Delta)$ where $\Delta > 0$ and $\limsup_{t \to 0+} \frac{1}{t} \alpha(t) < \infty$.

Then f possesses an H-derivative at the point x_0 .

<u>Proof</u>. Let $\varepsilon > 0$ and $h \in X$ be given. There exists $\delta_h \in (0, \Delta)$ such that $\frac{1}{t} \alpha(t) < C$ for all $t \in (0, \delta_h)$ where $C = 1 + \limsup_{t \to 0+} \frac{1}{t} \alpha(t)$. Further, there is $\eta_h > 0$ such that $\varphi(h - k) < \frac{\varepsilon}{C}$ whenever $\| h - k \| < \eta_h$. Consequently, $\| \frac{1}{t} [f(x_0 + th) - f(x_0 + tk)] \| < \varepsilon$ whenever $t \in (0, \delta_h)$ and $\| h - k \| < \eta_h$ and hence our assertion is true by Theorem 1.

Now, let M be a neighbourhood of a point $x_0 \in X$ and suppose there is a Gâteaux variation $Vf(x, h)$ (i.e., a nonlinear G-differential - see [6],[9]) of a mapping $f : X \longrightarrow Y$ at all points of M . Consider the following three types of continuity of the mapping $Vf(x,h) : M \times X \rightarrow Y$:

(a) $Vf(x, h)$ is separately continuous at all points of $\{x_0\} \times X$,

(b) $Vf(x, h)$ is jointly continuous at all points of $\{x_0\} \times X$,

(c) $Vf(x, h)$ is jointly continuous at all points of $\{x_0\} \times X$
and, moreover, it is continuous in the variable x at the point x_0 uniformly with respect to $h \in X$ with $\| h \| \leq 1$.

It is easy to see, because of homogeneity of $Vf(x, h)$ in h, that (a),(b),(c) are equivalent to the following conditions (a′),(b′) and (c′), respectively:

(a′) $Vf(x, h)$ is continuous in x at the point x_0 for every fixed $h \in X$ and $Vf(x_0, h)$ is continuous at the point $h = 0$;

(b′) $Vf(x, h)$ is jointly continuous at the point $(x_0, 0)$;

(c′) $Vf(x, h)$ is continuous in x at x_0 uniformly with respect to $h \in X$ with $\|h\| \leq 1$ and $Vf(x_0, h)$ is continuous at the point $h = 0$.

According to the well-known theorem of Vainberg (see [9]), the condition (a′) is a sufficient one for the G-variation $Vf(x_0, h)$ to be linear and continuous in h; it means f possesses a G-derivative $f'(x_0)$ at x_0 and $Vf(x, h) = f'(x_0)h$ for all $h \in X$. Further, as it can be easily shown, the condition (c′) implies the mapping f has a Fréchet derivative at x_0. So, the conditions (a) and (c) imply the existence of a G- and F-derivative of f at x_0, respectively, and hence a natural question arises what is the meaning of the condition (b). The answer is given by

<u>Theorem 2</u>. Let x_0 be a point of X, $M \subset X$ a neighbourhood of x_0 and let $f : M \longrightarrow Y$ be a mapping having a G-variation $Vf(x, h)$ in M. Suppose $Vf(x, h) : M \times X \longrightarrow Y$ is continuous at the point $(x_0, 0)$.

Then f possesses an H-derivative $f'(x_0)$ at the point x_0 and $Vf(x_0, h) = f'(x_0)h$ for all $h \in X$.

Proof. We can assume that the set M is convex and open. Let h_0 be an arbitrary point of X, let $\varepsilon > 0$ be given.

Continuity of $Vf(x, h)$ implies a G-derivative $f'(x_0)$ of f at x_0 exists (see [9]). Moreover, there are $\mu > 0$ and $\eta > 0$ such that

$$x_0 + x \in M \qquad \text{and}$$

$$\|Vf(x_0 + x, y)\| = \|Vf(x_0 + x, y) - Vf(x_0, 0)\| \leq \varepsilon \tag{8}$$

whenever $\|x\| < \mu$ and $\|y\| < \eta$, $x, y \in X$. Set $\delta = \mu(\eta + \|h_0\|)^{-1}$; then all points $x_0 + th_0$, $x_0 + th$ and $x_0 + th + \vartheta t(h - h_0)$ belong to M if $t \in (0, \delta)$, $\vartheta \in [0, 1]$ and $\|h - h_0\| < \eta$. By the mean-value theorem, there is $\vartheta_{t,h} \in [0, 1]$ for every $t \in (0, \delta)$ and $h \in X$, $\|h - h_0\| < \eta$ such that

$$\left\| \frac{1}{t} [f(x_0 + th_0) - f(x_0 + th)] \right\| \leq \tag{9}$$

$$\leq \left\| \frac{1}{t} Vf(x_0 + th_0 + \vartheta_{t,h}(th - th_0), th - th_0) \right\| =$$

$$= \| Vf(x_0 + th_0 + \vartheta_{t,h} t(h - h_0), h - h_0) \| .$$

It follows now from (8) and (9) that for every given $\varepsilon > 0$ and $h_0 \in X$, there are $\delta > 0$ and $\eta > 0$ such that

$$\left\| \frac{1}{t} [f(x_0 + th_0) - f(x_0 + th)] \right\| \leq \varepsilon$$

for all $t \in (0, \delta)$ and $h \in X$ with $\|h - h_0\| < \eta$. Hence $f'(x_0)$ is a H-derivative of f at x_0 by our Theorem 1.

Remark. Let $g(t, u)$ be a real function on $G \times \times R$ ($G \subset R_n$) satisfying the Carathéodory conditions and having a partial derivative $g'_n(t, u)$, let $g'_n(t, u)$ be bounded and continuous in the variable u. M.M. Vainberg showed in [10] that a mapping $x(t) \longrightarrow g(t, x(t))$ from $L_2(G)$ into $L_2(G)$ possesses a jointly continuous G-differential $Dg(x, h)(t) = g'_n(t, x(t))h(t)$ $(x, h \in L_2(G))$ but it is nowhere Fréchet differentiable in $L_2(G)$.

Another example of an operator satisfying all assumptions of our theorem and being not Fréchet differentiable, was given by Alexiewicz and Orlicz in [1].

3. Let G and G' be bounded subsets of n-dimensional and m-dimensional Euclidean spaces R_n and R_m, respectively, let p and q be arbitrary numbers, $1 \leq \leq p, q \leq \infty$. Denote by $M_q(G')$ and $N_p(G)$ the sets of all real functions $x(s, t)$ on $G' \times G$ such that

$$\|x(s, t)\|_{M_q} = \left\| \int_G x(s, t)\, dt \right\|_{L_q} < \infty$$

or

$$\|x(s, t)\|_{N_p} = \left\| \sup_{s \in G'} \operatorname{ess} |x(s, t)| \right\|_{L_p} < \infty ,$$

respectively; here $\|\cdot\|_{L_p}$ denotes an ordinary norm in the Lebesgue space $L_p(G)$ and $\|\cdot\|_{L_q}$ a norm in $L_q(G')$. It is a well-known fact (see [5]) that $M_q(G')$ and $N_p(G)$ are linear spaces, $\|\cdot\|_{M_q}$ and $\|\cdot\|_{N_p}$ are norms in $M_q(G')$ and $N_p(G)$, respectively, and $M_q(G')$ and $N_p(G)$ with these norms are Banach spaces.

Now, let $K(s,t,u)$ be a real function on $G'\times G\times R$ satisfying the Carathéodory conditions; i.e. $K(s,t,u)$ is measurable in s for almost every $t\in G$ and all $u\in R$, measurable in t for a.e. $s\in G'$ and all $u\in R$ and continuous in u for a.e. $s\in G'$ and a.e. $t\in G$. An operator $\mathbb{K}$ defined by the formula

$$\mathbb{K}x(s)=\int_G K(s,t,x(t))\,dt \quad (s\in G')$$

(for measurable functions $x(t)$ on G) is called the Urysohn operator (defined by $K(s,t,u)$).

We suppose in the proposition below that a partial derivative $K'_u(s,t,u)$ of the function $K(s,t,u)$ exists in $G'\times G\times R$ and that both functions satisfy the Carathéodory conditions.

<u>Proposition</u>. Let $1\le p,\ q\le\infty$, $x_0\in L_p(G)$, let $U\subset L_p(G)$ be an open neighbourhood of zero, $M=x_0+U$ and denote $\tilde{U}=\{x(s,t)\in N_p(G):\ \operatorname{supess}_{s\in G'}|x(s,t)|\in U\}$.

Let $K, K'_u, \mathbb{K}$ be as above and define an operator $\mathcal{K}$ by

$$\mathcal{K}(x,h)(s,t)=K'_u(s,t,x_0(t)+x(s,t))\,h(t)\ (s\in G', t\in G)$$

for $x \in \mathcal{N}_p(G)$ and $h \in L_p(G)$. Suppose $\mathcal{K}$ maps $\tilde{\mathcal{U}} \times L_p(G)$ into $\mathcal{M}_q(G')$ is continuous at the point $(0,0)$ and $\mathcal{K}(x,h)$ is continuous in the variable x on $\tilde{\mathcal{U}}$ for every fixed $h \in \mathcal{U}$. Let $\mathbb{K}x_0 \in L_q(G')$. Then $\mathbb{K} : \mathcal{M} \longrightarrow L_q(G')$ and there is an H-derivative $\mathbb{K}'(x_0) : L_p(G) \longrightarrow L_q(G')$ of the operator $\mathbb{K}$ at the point x_0. Moreover,

$$\mathbb{K}'(x_0)h(s) = \int_G K'_u(s,t,x_0(t))h(t)\,dt \qquad (s \in G')$$

for all $h \in L_p(G)$.

__Proof.__ By the mean-value theorem, there is a measurable (see [5]) function $\vartheta(s,t)$, $0 \le \vartheta(s,t) \le 1$, such that

$$\left\| \int_G K(s,t,x_0(t)+x(t))\,dt \right\|_{L_q} \le$$

$$\le \left\| \int_G K(s,t,x_0(t))\,dt \right\|_{L_q} + \left\| \int_G K'_u(s,t,x_0(t)+\vartheta(s,t)x(t))x(t)\,dt \right\|_{L_q} =$$

$$= \| \mathbb{K}x_0 \|_{L_q} + \| \mathcal{K}(\vartheta x, x) \|_{M_q}$$

for all $x \in \mathcal{U}$. The last term is finite for these x and so $\mathbb{K}(\mathcal{M}) \subset L_q(G')$.

It follows from the mean-value theorem again that

$$\Big\| \frac{1}{\tau}[\mathbb{K}(x_0+x+\tau h)(s) - \mathbb{K}(x_0+x)(s)] -$$

$$- \int_G K'_u(s,t,x_0(t)+x(t))h(t)\,dt \Big\|_{L_q} =$$

$$= \Big\| \int_G [K'_u(s,t,x_0(t)+x(t)+\vartheta(s,t))h(t) -$$

$$- K'_u(s,t,x_0(t)+x(t))h(t)]\,dt \Big\|_{L_q} =$$

$$= \| \mathcal{K}(x+\vartheta\tau h, h) - \mathcal{K}(x,h)\|_{M_q}$$

for every $x \in \mathcal{U}$, $h \in L_p(G)$ and sufficiently small $\tau \neq 0$ $(0 \leq \vartheta(s,t) \leq 1)$. The last term converges to zero if $\tau \to 0$ because of homogeneity of $\mathcal{K}(x,h)$ in h and because $\|\vartheta\tau h\|_{N_p} \to 0$ in that case. Hence $\mathbb{K}$ possesses a G-differential at all points of M and

$$\mathbb{K}'(x)h(s) = \int_G K'_u(s,t,x(t))\,h(t)\,dt \qquad (s \in G')$$

for $x \in M$ and $h \in L_p(G)$. Moreover,

$$\|\mathbb{K}'(x)h\|_{L_q} = \|\mathcal{K}(x-x_0,h)\|_{M_q}$$

for these x, h and so the G-differential is jointly continuous at the point $(x_0, 0)$. Now, Theorem 2 implies that $\mathbb{K}'(x_0)$ is an H-derivative of $\mathbb{K}$ at x_0.

Remark. Our restriction to the case of a Lebesgue measure is not essential and we can state the proposition for more general spaces of the type of $L_p(G, \Sigma, \mu)$.

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