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HYPER-EXTENSIONS OF σ-ALGEBRAS

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Abstract: A new operation "measurably discrete" union is introduced, and for any σ-algebra two σ-algebras $h\mathcal{a}$ and $H\mathcal{a}$ $(\supset h\mathcal{a})$ are introduced and the extension of two-valued measures is investigated. For an extension theorem for real valued measures two similar hyper-extensions $h_{\aleph}\mathcal{a}$ and $H_{\aleph}\mathcal{a}$ are introduced. The results were discussed in my 1970-71 seminar, and they were announced in my talk at the 3rd Prague Symposium in Topology and its Applications.

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1. Discrete families wrt a collection.

Let $\mathcal{M}$ be a collection of sets. A family $\{M_a \mid a \in A\}$ in $\mathcal{M}$ is called completely additive in $\mathcal{M}$ (or completely $\mathcal{M}$-additive) if the union of each subfamily of $\{M_a\}$ belongs to $\mathcal{M}$. A family of sets $\{X_a \mid a \in X\}$ is called $\mathcal{M}$-discrete if there exists a completely $\mathcal{M}$-additive disjoint family $\{M_a\}$ such that $M_a \supset X_a$ for each a .

Evidently, if $\{X_a\}$ is $\mathfrak{M}$-discrete, and if $Y_a \subset$ $\subset X_a$ for each a , then $\{Y_a\}$ is $\mathfrak{M}$-discrete. The definitions are given in full generality, although our results will merely concern the case when $\mathfrak{M}$ is a σ-algebra.

A collection $\mathfrak{M}$ is called separable if each $\mathfrak{M}$-discrete family is countable. Recall that a collection $\mathfrak{M}$ satisfies the CCC (countable chain condition) if each disjoint family in $\mathfrak{M}$ is countable. Hence CCC implies separability, and the converse is true if $\mathfrak{M}$ is completely additive. By a non-trivial Theorem 1 in Frolík [1] every analytic σ-algebra is separable. If $2^{\aleph_0} = \aleph_1$ then every countably generated σ-algebra is separable (because if there exists a completely $\mathfrak{M}$-additive disjoint family of cardinal m , then the cardinal of $\mathfrak{M}$ is greater or equal to 2^m) .

There is a very interesting result by F. Hansell [1] which says that if X is a completely metrizable space then every $Ba\,X$ -discrete family is σ-discretely decomposable, see Frolík [3] or [5] .

<u>Definition</u>. A collection $\mathfrak{N}$ is called $D_{\mathfrak{M}}$-closed if every $\mathfrak{M}$-discrete family of non-measurable cardinal in $\mathfrak{N}$ is completely additive in $\mathfrak{N}$. We denote by $D_{\mathfrak{M}}\mathfrak{N}$ the collection which consists of the unions of $\mathfrak{M}$-discrete families of non-measurable cardinal ranging in $\mathfrak{N}$. If $\mathfrak{N} = \mathfrak{M}$ then we write simply D instead of $D_{\mathfrak{M}}$.

<u>Remark</u>. F. Hansell [1],[2] studied extension of the

Borel σ-field in a metric space by "topologically" discrete unions. In a complete metric space both operations are equivalent by the Hansell's result referred to above.

For example, every separable σ-algebra is $\mathcal{D}$-closed.

2. Hyper-rocks.

If $\mathcal{N}$ is a collection of sets then $\rho\mathcal{N}$ ($\alpha\mathcal{N}$, respectively) denotes the smallest countably multiplicative and countably additive (σ-algebra, respectively) which contains $\mathcal{N}$.

Definition. Let $\mathcal{M}$ be a collection of sets. For any collection $\mathcal{N}$ let

$\rho_{\mathcal{h}_{\mathcal{M}}}\mathcal{N}$ be the smallest $\mathcal{F} \supset \mathcal{N}$ such that
$$\rho\mathcal{F} = \mathcal{D}_{\mathcal{M}}\mathcal{F} = \mathcal{F},$$
and let

$\alpha_{\mathcal{h}_{\mathcal{M}}}\mathcal{N}$ be the smallest $\mathcal{F} \supset \mathcal{N}$ such that
$$\alpha\mathcal{F} = \mathcal{D}_{\mathcal{M}}\mathcal{F} = \mathcal{F}.$$

If $\mathcal{N}$ is a collection of sets denote by $\rho_H\mathcal{N}$ the smallest $\mathcal{F} \supset \mathcal{N}$ such that $\rho\mathcal{F} = \mathcal{D}\mathcal{F} = \mathcal{F}$, and denote by

$\alpha_H\mathcal{N}$ the smallest $\mathcal{F} \supset \mathcal{N}$ such that
$$\alpha\mathcal{F} = \mathcal{D}\mathcal{F} = \mathcal{F}.$$

The elements of $\rho_{\mathcal{h}_{\mathcal{M}}}\mathcal{N}$ are called the $\mathcal{M}$-hyper-rocks over $\mathcal{N}$, the elements of $\rho_H\mathcal{N}$ are called the hyper-rocks over $\mathcal{N}$.

Finally we define $\rho_{\mathcal{h}}$ and $\alpha_{\mathcal{h}}$ as follows:

$$\rho_{\mathfrak{h}} \mathcal{N} = \rho_{\mathfrak{h}_\rho \mathcal{N}} \quad ,$$

$$\alpha_{\mathfrak{h}} \mathcal{N} = \alpha_{\mathfrak{h}_\alpha \mathcal{N}} \quad .$$

Clearly:

$$\alpha_{\mathfrak{h}} \mathcal{M} = \mathcal{M} \text{ if and only if } \alpha_H = \mathcal{M} = \mathcal{M} \ ;$$

and

$$\rho_{\mathfrak{h}} \mathcal{M} = \mathcal{M} \text{ if and only if } \rho_H = \mathcal{M} = \mathcal{M} \ .$$

It is well known that if $\mathcal{F}$ is a cover of X, and if

$$compl\ \mathcal{F} \subset \rho\ \mathcal{F} \ ,$$

then

$$\rho\ \mathcal{F} = \alpha\ \mathcal{F} \ .$$

Similar results hold for hyper extensions.

<u>Theorem A</u>. Let $\mathcal{F}$ be a cover of X. If

$$compl_X\ \mathcal{F} \subset \rho_H\ \mathcal{F}$$

then

$$\rho_H\ \mathcal{F} = \alpha_H\ \mathcal{F}$$

(and the converse is obvious).

<u>Proof</u>. Consider the collection

$$\mathcal{G} = E\{M \mid M \in \rho_H\ \mathcal{F},\ X - M \in \rho_H\ \mathcal{F}\} \ .$$

We have $\mathcal{F} \subset \mathcal{G} \subset \alpha_H\ \mathcal{F}$, and $\mathcal{G}$ is complemented, and hence it is enough to show that

$$\sigma \mathcal{G} \subset \mathcal{G} \supset D\mathcal{G} \ .$$

If $\mathcal{U}$ is the union of a sequence $\{\mathcal{U}_n\}$ in $\mathcal{G}$, then $X - \mathcal{U}$ is the intersection of the sequence $\{X - \mathcal{U}_n\}$ in $\mathcal{G}$, and hence both $\mathcal{U}$ and $X - \mathcal{U}$ belong to $\rho_H\ \mathcal{F}$, and hence $\mathcal{U} \in \mathcal{G}$.

Now let $\mathcal{U}$ be the union of a $\mathcal{G}$-discrete family $\{\mathcal{U}_a \mid a \in A\}$ in $\mathcal{G}$. There exists a completely $\mathcal{G}$-additive family $\{G_a\}$ with $G_a \supset \mathcal{U}_a$ for each a . We may and shall assume that $\{G_a\}$ is a cover of X . Since $\{G_a\}$ is completely $\wp_H \mathcal{F}$ -additive, and both $\{\mathcal{U}_a\}$ and $\{G_a - \mathcal{U}_a\}$ range in $\wp_H \mathcal{F}$, the two sets $\mathcal{U}$ and $X - \mathcal{U}$ belong to $\wp_H \mathcal{F}$, and hence, by definition of $\mathcal{G}$, $\mathcal{U}$ belongs to $\mathcal{G}$.

<u>Theorem B</u>. Let $\mathcal{F}$ be a cover of X . If

$$compl_X \mathcal{F} \subset \wp_{\mathfrak{h}} \mathcal{F}$$

then

$$\wp_{\mathfrak{h}} \mathcal{F} = \alpha_{\mathfrak{h}} \mathcal{F}$$

(and the converse is obvious).

<u>Proof</u>. Show that

$$compl_X \wp \mathcal{F} \subset \wp_{\mathfrak{h}} \mathcal{F}$$

and proceed as in the proof of Theorem A

<u>Corollary</u>. If $\mathcal{F}$ is an algebra then

$$\wp \mathcal{F} = \alpha \mathcal{F}, \quad \wp_{\mathfrak{h}} \mathcal{F} = \alpha_{\mathfrak{h}} \mathcal{F}, \quad \wp_H \mathcal{F} = \alpha_H \mathcal{F} \quad .$$

<u>Remark</u>. It is obvious that Theorems A and B hold for the analogous concepts which we get by deleting the assumptions of non-measurability of discrete unions. On the other hand, the assumption of non-measurability of the admitted unions is essential in what follows.

3. Extensions of maximal filters with CIP.

Here we prove one of the two our main results. By a paving of a set X we mean a cover of X which is closed under finite unions and finite intersections.

Theorem C. Let $\mathcal{F}$ be a paving of X, and let $\mathcal{F}' = \varphi_H \mathcal{F}$. If Φ is a maximal $\mathcal{F}$-filter with CIP, then

$$\Psi = E\{M \mid M \in \mathcal{F}', \; M \supset N \in \sigma \Phi \quad \text{for some } N\}$$

is a maximal filter in $\mathcal{F}'$, and hence

$$\Psi = E\{M \mid M \in \mathcal{F}', \; M \cap [\sigma \Phi] \quad \text{is centred}\},$$

in particular, Ψ is the only maximal $\mathcal{F}'$-filter with

$$\Psi \cap \mathcal{F} = \Phi \ .$$

Proof. Let Φ^* be the filter in $\exp X$ which has Φ for a basis. Consider the collection $\mathcal{G}$ of all $G \in \mathcal{F}'$ such that

$$\text{if } G \cap [\sigma \Phi] \text{ is centred then } G \in \Phi^* .$$

We shall prove that $\varphi_H \mathcal{G} \subset \mathcal{G}$; since $\mathcal{G} \supset \mathcal{F}$ it would follow $\mathcal{G} = \mathcal{F}$!

Assume that $M = \bigcap\{M_n\}$ where $\{M_n\}$ is a sequence in $\mathcal{G}$. If M meets each element of $\sigma\Phi$ then so does each M_n, and hence we can choose a sequence $\{F_n\}$ in $\sigma\Phi$ such that $F_n \subset M_n$; now the intersection F of $\{F_n\}$ belongs to $\sigma\Phi$ and is contained in M. Thus $M \in \mathcal{G}$.

Now let $M = \bigcup\{M_n\}$ where $\{M_n\}$ is a sequence in $\mathcal{G}$. If M meets each element of $\sigma\Phi$, then some M_n meets each element of $\sigma\Phi$; indeed, if the converse were true then there would be a sequence $\{F_n\}$ in $\sigma\Phi$ with $M_n \cap F_n \neq \emptyset$, and hence $F = \bigcap\{F_n\} \in \sigma\Phi$ would be disjoint to M. If M_n meets each element of $\sigma\Phi$ then $M_n \in \Phi^*$, and hence $M \in \Phi^*$.

Finally, let M be the union of a $\mathcal{G}$-discrete non-measurable family $\{M_a\}$ in $\mathcal{G}$, and let M meet each element of $\sigma\Phi$. Choose a disjoint completely $\mathcal{G}$-additive family $\{G_a\}$ in $\mathcal{G}$ such that $G_a \supset M_a$ for each a. Consider the collection $\mathcal{A}$ of subsets B of A such that

$$G_B = \bigcup\{G_a \mid a \in B\} \in \Phi^* .$$

Since $\{G_B\}$ ranges in $\mathcal{G}$, if $C \subset A$ then either $C \in \mathcal{A}$ or $A - C \in \mathcal{A}$. It follows that $\mathcal{A}$ is a maximal filter with CIP in $\exp A$. Since the cardinal of A is non-measurable, some $(a) \in \mathcal{A}$ where $a \in A$. Thus $G_a \in \Phi^*$. Hence $M_a = M \cap G_a$ meets each element of $\sigma\Phi$, and hence $M_a \in \Phi^*$ because $M_a \in \mathcal{G}$. This concludes the proof.

__Corollary__. Let $\mathcal{F}$ be a paving of X, and for each maximal $\mathcal{F}$-filter with CIP Φ let Φ' be the filter in $\mathcal{F}' = \wp_H \mathcal{F}$ which has Φ for its basis. Then $\{\Phi \to \Phi'\}$ is a bijection onto the set of all maximal $\mathcal{F}'$-filters with CIP.

__Remark__. This is a generalization of the result of Hayes [1] and Frolík [2] which we get by replacing $\wp_H$ by $\wp$. It should be remarked that the result extends to hyper-Souslin sets (the questions of taking countable intersections and countable unions are replaced by the Souslin operation).

It seems to be useful to state explicitly the following result which was proved in the proof of Theorem C.

Lemma A. Let $\mathcal{H}$ be a paving of X, and let Ψ be a maximal $\mathcal{H}$-filter with CIP. If $H \in \mathcal{H}$ is the union of an $\mathcal{H}$-discrete non-measurable family $\{H_a\}$ in $\mathcal{H}$ then $H_a \in \Psi$ for some a.

4. Maximal filters based in a sub-paving.

Theorem D. Let $\mathcal{F}$ and $\mathcal{G} \supset \mathcal{F}$ be pavings of X, and let Ψ be a maximal, $\mathcal{G}$-filter with CIP. Let $\mathcal{F}' = \varphi_{\mathcal{R}\mathcal{G}} \mathcal{F}$. There exists a maximal $\mathcal{F}'$-filter with CIP Ψ' such that

$$\Psi' \cap \mathcal{F} = \Psi \cap \mathcal{F} \quad ;$$

in addition $\sigma \Phi$, where $\Phi = \Psi \cap \mathcal{F}$, is a basis for Ψ'.

Proof. By Theorem C we may and shall assume that $\mathcal{G}$ is D-closed and φ-closed. Then $\mathcal{F}' \subset \mathcal{G}$. Denote by Φ^* the filter in $\exp X$ which has $\sigma\Phi$ for its basis. Consider the collection $\mathcal{L}$ of all M in $\mathcal{F}'$ such that

if $M \in \Psi$ then $M \in \Phi^*$.

It is enough to show that $\varphi \mathcal{L} \subset \mathcal{L}$ and $D_{\mathcal{G}} \mathcal{L} \subset \mathcal{L}$. The proof of the former relation follows (verbatim) the proof of a similar relation in § 3. The proof of the latter relation follows the pattern of the proof of Theorem C, however we sketch the proof. Assume that L is the union of a $\mathcal{G}$-discrete family $\{L_\alpha\}$ in $\mathcal{L}$, and let G be the union of a corresponding completely $\mathcal{G}$-additive family $\{G_\alpha\}$ which dominates $\{L_\alpha\}$. If $L \notin \Psi$ then $L \in \mathcal{L}$ by definition. If $L \in \Psi$ then $G \in \Psi$, and by Lemma A $G_\alpha \in \Psi$ for some α. Since $M \in \Psi$, $M_\alpha = G_\alpha \cap M \in \Psi$, and since $M_\alpha \in \mathcal{L}$, necessarily $M_\alpha \in \Phi^*$, and hence $M \in \Phi^*$.

The following corollary seems to be worth of stating.

Lemma B. Let $\mathcal{H}$ be a paving of X, and let Φ be a maximal $\mathcal{H}$-filter with CIP. Let H be the union of a family $\{H_\alpha\}$ in $\mathcal{H}$. If there exists a paving $\mathcal{K} \supset \mathcal{H}$ of X such that

(a) $\{H_\alpha\}$ is $\mathcal{K}$-discrete,

and

(b) Φ extends to a maximal $\mathcal{K}$-filter with CIP, then $H_\alpha \in \Phi$ for some α.

5. Applications. For definitions see Frolík [3] or [4].

Theorem E. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be pavings of X, $\mathcal{F} \subset \mathcal{G} \subset \mathcal{H}$, and let $\mathcal{H}$ be φ_H-closed, and let

$$\mathcal{G} = \varphi_{h\mathcal{H}} \mathcal{F} .$$

Then:

a) If $\mathcal{G} = \mathcal{H}$ then $\mathcal{F}$ is mcximal-complete if and only if $\mathcal{H}$ is maximal-complete. In other words, $\mathcal{F}$ is maximal-complete if and only if $\varphi_H \mathcal{F}$ is maximal-complete.

b) If $\mathcal{F}$ is maximal-complete, and if Ψ is a maximal $\mathcal{H}$-filter with CIP, then $\bigcap \Psi \cap \mathcal{G} \neq \emptyset$.

<u>Remark</u>. In the usual terminology in measure theory one may restate the results as follows (by a measure we shall mean a two-valued σ-additive measure). If μ is a measure on $\mathcal{F}$ (now $\mathcal{F}$ is an algebra) then μ extends uniquely to a measure μ' on $\alpha_H \mathcal{F}$, and μ' is $\delta\mathcal{F}$-regular. If μ' is a measure on $\mathcal{H}$, and if $\mathcal{F} \subset \mathcal{H}$ is a paving, then the restriction μ of μ' to $\alpha_{\mathfrak{R}\mathcal{H}} \mathcal{F}$ is $\delta\mathcal{F}$-regular.

6. <u>Remarks.</u> Let $\mathcal{F}$ and $\mathcal{G} \supset \mathcal{F}$ be pavings of X. We propose to call $\mathcal{F}$ <u>descriptive in</u> $\mathcal{G}$ if $\varphi_H \mathcal{F} \supset \mathcal{G}$, and to call $\mathcal{F}$ weakly descriptive in $\mathcal{G}$ if

$\varphi_{\mathfrak{R}_{\varphi_H \mathcal{G}}} \mathcal{F} \supset \mathcal{G}$.

Thus, $\mathcal{F}$ is descriptive or weakly descriptive in $\mathcal{G}$ if and only if $\mathcal{F}$ has the corresponding property in $\varphi_H \mathcal{G}$.

If X is a topological space then the following choices for $\mathcal{F}$ and $\mathcal{G}$ seem to be interesting:

a) $\mathcal{F} = \mathrm{Ba}\, X$, $\mathcal{G} = \mathrm{B\sigma}\, X$.

b) $\mathcal{F} =$ closed X, $\mathcal{G} = \mathrm{B\sigma}\, X$.

c) $\mathcal{F} =$ compact X, $\mathcal{G} = \mathrm{B\sigma}\, X$.

d) $\mathcal{F} = \mathrm{Ba}\, X \cap$ compact X, $\mathcal{G} = \mathrm{Ba}\, X$.

These cases will be treated elsewhere. The case a) is studied in Frolík [6] and Hager [1].

7. Extensions of real valued measures.

In § 1 we made cardinal restriction on taking discrete unions to have Theorems B, C and D and E on extension of maximal filters with CIP, or equivalently on extension of two valued measures. From the point of view of descriptive theory of sets the concepts we obtain first by supressing the cardinal restriction on discrete unions are of interest. In this section we discuss similar theorems for extension of real valued measures. Then further cardinal restrictions on discrete unions are necessary.

Call a cardinal $\aleph$ real non-measurable if there is no real valued non-negative σ-additive measure μ on $\exp X$ where X is a set of cardinal $\aleph$ such that $\mu(x) = 0$ for each $x \in X$, and $\mu X = 1$. Hence, if $\aleph$ is real non-measurable then it is non-measurable. It was proved by Ulam that $\aleph_1$ is real non-measurable, and if some $\aleph$ is non-measurable and real measurable then $\exp \aleph_0$ is real measurable.

We noted in § 2 that Theorems A and B hold for the concepts without cardinal restrictions on the discrete unions (but not Theorems C, D, and E), It is easy to check that Theorems A and B hold if the operation of taking discrete unions is restricted to families of real non-measurable cardinal; the resulting theorems are called Theorem Ar and Theorem Br. The notation with cardinal restriction to real non-measurable unions is obtained by subscript κ to $\mathfrak{D}, \mathfrak{h}$ and

H .

<u>Remark</u>. In general it would be much convenient to have $\mathcal{D}, \mathcal{h}$ and $\mathcal{H}$ for the setting without any cardinal restriction on the unions, $\mathcal{D}_2$, $\mathcal{h}_2$ and $\mathcal{H}_2$ for the setting with non-measurable unions, and $\mathcal{D}_\kappa$, $\mathcal{h}_\kappa$ and $\mathcal{H}_\kappa$ for the setting with real non-measurable unions. This is the notation we shall use elsewhere.

By a probability on an σ -algebra we mean a non-negative real valued measure such that the measure of the whole spaces is 1 .

<u>Theorem F</u>. Let $\mathcal{A}$ be a σ -algebra on a set X , and let $\mathcal{B} = \alpha_{H_\kappa} \mathcal{A}$. If μ is a probability on $\mathcal{A}$, then there exists a unique probability ν on $\mathcal{B}$ which extends ν . In addition, for each B in $\mathcal{B}$ there exist A_1, A_2 in $\mathcal{A}$ such that

$$A_1 \subset B \subset A_2 \quad ,$$

and

$$\mu A_1 = \mu A_2 \ (= \nu B) \quad .$$

<u>Corollary</u>. If there exists a probability μ on $\mathcal{A}$ such that $\langle X, \mathcal{A}, \mu \rangle$ is a complete probability space (i.e. $\mu A = 0$ implies $\exp A \subset \mathcal{A}$), then $\alpha_{H_\kappa} \mathcal{A} = \mathcal{A}$.

<u>Remark</u>. For a σ -algebra $\mathcal{A}$ let $\cup(\mathcal{A})$ stand for the σ -algebra of universally measurable sets wrt $\mathcal{A}$, i.e. the intersection of the completions of $\mathcal{A}$ wrt probabilities on $\mathcal{A}$. Theorem F says that

$$\alpha_{H_\kappa} \mathcal{A} \subset \cup(\mathcal{A}) \quad .$$

Certainly the inclusion is strict in general. In fact the elements of our extensions are as respectable sets in non-separable case as Borel sets in separable case are. If $\mathcal{A}$ is separable, then $\mathcal{D}\mathcal{A} = \mathcal{D}_2\mathcal{A} = \mathcal{D}_\aleph\mathcal{A} = \mathcal{A}$, and hence the extensions are trivial.

Proof of Theorem F. Denote by $\mathcal{C}$ the completion of $\mathcal{A}$ wrt μ and let π be the corresponding unique extension of μ . Thus $\mathcal{C}$ is the set of all $C \subset X$ such that $\mu A_1 = \mu A_2$ for some $A_i \in \mathcal{A}$, $A_1 \subset C \subset A_2$. Clearly $\mathcal{C}$ is a σ -algebra. We must show that

$$\mathcal{B} \subset \mathcal{C} .$$

It is enough to show that $\mathcal{D}_\aleph \mathcal{C} \subset \mathcal{C}$. Assume that C is the union of a $\mathcal{C}$ -discrete family $\{C_a \mid a \in A\}$, and assume that the cardinal of A is real non-measurable. Let

$$A' = \{a \mid \pi C_a \neq 0\} , \quad A'' = A - A' .$$

The set A' is countable by a standard argument, and the set A'' is real non-measurable because A is. Thus

$$\pi \bigcup \{C_a \mid a \in A''\} = 0 .$$

Take A_{1a}, A_{2a} in $\mathcal{A}$, $a \in A'$, such that

$$A_{1a} \subset C_a \subset A_{2a}$$

and

$$\mu A_{1a} = \mu A_{2a} ,$$

and take $A^* \in \mathcal{A}$ such that $A^* \supset \bigcup\{C_a \mid a \in A''\}, \mu A^* = 0$.

Put

$$A_1 = \bigcup\{A_{1a} \mid a \in A'\} ,$$

$$A_2 = A^* \cup \bigcup \{A_{2a} \mid a \in A'\} .$$

Clearly $A_1 \subset C \subset A_2$, $\mu A_1 = \mu A_2$. This concludes the proof.

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