Stanislav Palúch

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Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 2, 339--359

Persistent URL: `http://dml.cz/dmlcz/105495`

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14,2 (1973)

THE LATTICES OF NUMERATIONS OF THEORIES CONTAINING PEANO'S ARITHMETIC

Stanislav PALÚCH, Žilina

Abstract: Studying consistency statements for an arithmetic A one has to decide whether one considers (a) numerations or bi-numerations, (b) PR-formulas or RE-formulas, (c) a particular axiomatization of A or all equivalent axiomatizations. This yields various structures of numerations; all are lattices and have similar properties.

Key words: arithmetization, numeration, bi-numeration, lattice.

AMS: 02G99 Ref. Ž.: 2.669

Introduction. In a theory T containing the Peano's arithmetic P, many metamathematical notions can be described, i.e. numerated or bi-numerated. Some from them are for example the relation " φ is an axiom of the axiomatic system $\langle L, A \rangle$ ", the relation $\mathcal{P}rf_A(\varphi, d)$ meaning " d is the code of a sequence which is the proof of the formula φ in $\langle L, A \rangle$ ", the relation $\mathcal{P}r_T(\varphi)$ meaning "the formula φ is provable in the theory T ", $Fm_{K_0}(\varphi)$ meaning "the formula φ is a formula of the language K_0 " etc. For a bi-numeration α of some axiomatization of a theory T , we can construct a formula $\mathcal{P}rf_{\alpha}(x, y)$ which is a bi-numeration of the relation $\mathcal{P}rf_{T}(\varphi, d)$ in T , a formula $\mathcal{P}r_{\alpha}(x)$ which is a numeration of the relation $\mathcal{P}r_{T}(\varphi)$, a formula Con_{α} expressing formal consistency of T etc.

For two different bi-numerations α_1 , α_2 of an axiomatization A of the theory T , we need not have $T \vdash \alpha_1(x) \equiv \alpha_2(x)$; we can even find bi-numerations α_1 , α_2 for which $T \nvdash \alpha_1(x) \longrightarrow \alpha_2(x)$. On the basis of this fact we can construct - on any set Θ of some numerations or bi-numerations of the theory T in itself - an ordering $\leq_T$ defined as follows: $\alpha \leq_T \beta$ iff $T \vdash Con_{\beta} \rightarrow Con_{\alpha}$. The equivalence $\equiv_T$ is defined as follows: $\alpha \equiv_T \beta$ iff $\alpha \leq_T \beta$ and $\beta \leq_T \alpha$. Let us denote by $\langle \Theta \rangle$ the decomposition of the set Θ into equivalence classes w.r.t. $\equiv_T$. We define the following relation $\leq_T$ on the set $\langle \Theta \rangle$: $[\alpha] \leq_T [\beta]$ iff $\alpha \leq_T \beta$, where $[\alpha]$ is the class of $\langle \Theta \rangle$ such that $\alpha \in [\alpha]$. This structure, where Θ was the set of all PR-bi-numerations of one fixed axiomatization of a theory T satisfying certain conditions, was studied by M. Hájková in [2]. She has proved that $(\langle \Theta \rangle, \leq_T)$ is a lattice with various interesting properties.

The results of [2] seem to support the conjecture that there is no natural bi-numeration of the Peano's arithmetic P in the following sense: In the lattice of all PR-bi-numerations of a primitive recursive aximatization of P , no element is Σ_1-definable and the hypothesis is that no element is definable.

The class of PR-bi-numerations can be considered as the class of reasonable (simplest) bi-numerations. But it is not necessary to restrict ourselves to this particular case; there are other reasonable possibilities. We can get them by altering the following fundamental parameters:

1. The type of formalization. We can consider the set Θ as the set of all bi-numerations or as the set of all numerations.

2. The type of formulas. We admit two fundamental types of formulas corresponding syntactically to primitive recursive sets and recursively enumerable sets respectively, namely PR-formulas and RE-formulas.

3. The number of formalized axiomatizations. We can consider Θ as the set of formalizations of one fixed axiomatization of a theory T or as the set of formalizations of all axiomatizations of a theory T. We restrict ourselves to recursively enumerable axiomatizations.

Each of the mentioned parameters can take two different values. Thus we get 8 combinations and every combination defines some set of formalizations of the theory T in itself. In this paper, we consider all these sets with the ordering $\leq_T$. We show that all structures have very similar properties, some from them are even isomorphic.

The reader is expected to be familiar with the Feferman's paper [1] (§§ 2 - 5 and a part of § 7) and, in particular, with the paper [2] of M. Hájková; this work is very closely connected with [2].

I am thankful to P. Hájek for his kind encouragement and help with the organization of the results and translation of the present paper.

§ 1. Definitions and statements

An axiomatic system is a pair $\mathfrak{A} = \langle L, A \rangle$, where L is a language and A a subset of the set of all formulas of L. We say that a formula φ is provable in $\mathfrak{A}$ if it is provable from the set $Ax_L \cup A$ (where Ax_L is the set of all logical axioms in the language L, see [1]) by means of predicate calculus. A theory T is a pair $\langle L, B \rangle$ where L is a language, $B \subseteq Fm_L$ (Fm_L is the set of all formulas of the language L) and B is closed w.r.t. provability, i.e. $B = \mathcal{P}r_B$. Every set of formulas $A \subseteq Fm_L$ such that $B = \mathcal{P}r_A$ will be called an axiomatization of T. We shall say that a formula φ is provable in T if $\varphi \in B$. In this case we shall write $\mathcal{P}r_B(\varphi)$ or $\mathcal{P}r_T(\varphi)$ or $T \vdash \varphi$. It is easily seen that every axiomatic system $\mathfrak{A} = \langle L, A \rangle$ defines a theory $T = \langle L, \mathcal{P}r_A \rangle$.

<u>Convention</u>. We shall write

$$\begin{array}{llll} T \vdash \varphi_1 \rightarrow \varphi_2 & \quad \text{resp.} & T \vdash \varphi_1 \equiv \varphi_2 & , \\ T \vdash \quad\;\; \rightarrow \varphi_3 & & T \vdash \quad\;\; \equiv \varphi_3 & , \\ \vdots & & \vdots & \\ T \vdash \quad\;\; \rightarrow \varphi_m & & T \vdash \quad\;\; \equiv \varphi_m & , \end{array}$$

instead of

$$
\begin{array}{llll}
T \vdash \varphi_1 \longrightarrow \varphi_2 & \text{resp.} & T \vdash \varphi_1 \equiv \varphi_2 & , \\
T \vdash \varphi_2 \longrightarrow \varphi_3 & & T \vdash \varphi_2 \equiv \varphi_3 & , \\
\vdots & & \vdots & \\
T \vdash \varphi_{n-1} \longrightarrow \varphi_n & & T \vdash \varphi_{n-1} \equiv \varphi_n & .
\end{array}
$$

We shall write $Fm^*(x)$ instead of $Fm_L^{\omega}(x)$, in other cases we shall use the same notation as in [2].

1.1. <u>Definition</u>. Let Ω be an arbitrary set of formulas of a theory T and let A be an axiomatization of T. We define:

$\mathcal{B}in_T^{\infty}(\Omega) = \{\alpha \,;\, \alpha \in \Omega,\ \alpha$ is a bi-numeration of some axiomatization of T in $T\}$.

$\mathcal{N}um_T^{\infty}(\Omega) = \{\alpha \,;\, \alpha \in \Omega,\ \alpha$ is a numeration of some axiomatization of T in $T\}$.

$\mathcal{B}in_T^{A}(\Omega) = \{\alpha \,;\, \alpha \in \Omega,\ \alpha$ is a bi-numeration of the axiomatization A of T in $T\}$.

$\mathcal{N}um_T^{A}(\Omega) = \{\alpha \,;\, \alpha \in \Omega,\ \alpha$ is a numeration of the axiomatization A of T in $T\}$.

1.2. <u>Remark</u>. The sets defined in this definition can be empty. For example if A is an axiomatization of the Peano's arithmetic which is not primitive recursive then the set $\mathcal{B}in_T^{A}(PR)$ is empty because every PR-formula is a bi-numeration of a primitive recursive set in P.

1.3. Lemma. Let T be a consistent theory and let A be an axiomatization of T. Then

1) $\mathcal{B}in^{\infty}_{T}(\Omega) \subseteq \mathcal{N}um^{\infty}_{T}(\Omega)$,

2) $\mathcal{B}in^{A}_{T}(\Omega) \subseteq \mathcal{N}um^{A}_{T}(\Omega)$.

Proof: The statement is clear when we realize that every bi-numeration of A in T is a numeration of A in T if T is consistent.

1.4. Definition and lemma. Let Θ be an arbitrary set of bi-numerations or numerations of some axiomatizations of T in T. For $\alpha, \beta \in \Theta$ we define $\alpha \leq_T \beta$ iff $T \vdash \dot{Con}_{\beta} \rightarrow \dot{Con}_{\alpha}$, $\alpha \equiv_T \beta$ iff $\alpha \leq_T \beta$ and $\beta \leq_T \alpha$. The relation $\leq_T$ is reflexive and transitive - it is a quasi-ordering on Θ. The relation $\equiv_T$ is an equivalence on Θ. Denote by $\langle \Theta \rangle$ the decomposition of Θ into equivalence classes w.r.t. $\equiv_T$. For $\alpha \in \Theta$, $[\alpha]_{\langle\Theta\rangle}$ denotes the element of $\langle \Theta \rangle$ for which $\alpha \in [\alpha]_{\langle\Theta\rangle}$. It is clear that $[\alpha]_{\langle\Theta\rangle} = [\beta]_{\langle\Theta\rangle}$ iff $T \vdash \dot{Con}_{\alpha} \equiv \dot{Con}_{\beta}$. The relation $\leq_{T,\Theta}$ is defined on $\langle \Theta \rangle$ as follows: $[\alpha]_{\langle\Theta\rangle} \leq_{T,\Theta} [\beta]_{\langle\Theta\rangle}$ iff $\alpha \leq_T \beta$. It is defined correctly because if $[\alpha_1]_{\langle\Theta\rangle} = [\alpha]_{\langle\Theta\rangle}$, $[\beta_1]_{\langle\Theta\rangle} = [\beta]_{\langle\Theta\rangle}$ and $[\alpha]_{\langle\Theta\rangle} \leq_{T,\Theta} [\beta]_{\langle\Theta\rangle}$ then $T \vdash \dot{Con}_{\beta_1} \equiv \dot{Con}_{\beta}$, $T \vdash Con_{\alpha_1} \equiv Con_{\alpha}$, $T \vdash \dot{Con}_{\beta} \rightarrow \dot{Con}_{\alpha}$ and hence $T \vdash \dot{Con}_{\beta_1} \rightarrow \dot{Con}_{\alpha_1}$ which is $[\alpha_1]_{\langle\Theta\rangle} \leq_{T,\Theta} [\beta]_{\langle\Theta\rangle}$. Hence the definition of $\leq_{T,\Theta}$ is independent on the choice of representatives of the classes $[\alpha]_{\langle\theta\rangle}$, $[\beta]_{\langle\theta\rangle}$. The relation $\preccurlyeq_{T,\theta}$ is an ordering on $\langle\theta\rangle$. In the case when it will not cause any confusion we shall write only $\preccurlyeq_T$ instead of $\preccurlyeq_{T,\theta}$.

The following statement is a reformulation of [1], 4.13:

1.5. Theorem. Let T be an ω -consistent theory, $P \subseteq T$. Let α be an arbitrary RE-numeration of a recursively enumerable axiomatization A of T in T . Then we can construct primitive recursive axiomatization A_0 of T and its PR-numeration α_0 in T such that $T \vdash \mathcal{P}r_\alpha \equiv \mathcal{P}r_{\alpha_0}$.

This theorem will be the fundamental one for § 2.

§ 2. The lattice $\langle \mathcal{B}in_T^A(RE)\rangle$ of RE-bi-numerations

In this section we shall assume that

1) T is an ω -consistent theory,

2) T contains Peano's arithmetic P , i.e. $P \subseteq T$,

3) A is a recursive axiomatization of T .

Let us note that for T and A satisfying these presumptions $\mathcal{B}in_T^A(RE)$ is not empty, because every recursive set is RE-bi-numerable even in P .

2.1. Theorem. In $\langle \mathcal{B}in_T^A(RE)\rangle$ there is no maximal element.

Proof: Let $\alpha \in \mathcal{B}in_T^A(RE)$; then $T \nvdash \neg Con_\alpha$ because of ω -consistency of T . Let $S = T + Con_\alpha$. Clearly, S is consistent. For α we can construct a PR-bi-numeration α_0 of some axiomatization A_0 of T in T such that $T \vdash \mathcal{P}r_{\alpha}(x) \equiv \mathcal{P}r_{\alpha_0}(x)$. The formula $\beta(x) = \alpha_0(x) \cup x \approx \overline{Con_{\alpha}}$ is a PR-bi-numeration of S in S . Let ν_{β} be the Gödel's formula for β constructed by a diagonal construction (see 5.2 in [1]). S is consistent and so $S \not\vdash \nu_3$. By [1] $S \vdash \nu_{\beta} \equiv \neg \mathcal{P}r_{\beta}(\overline{\nu_3})$. Set

$$\alpha'(x) = \alpha(x) \vee Fm^*(x) \,\&\, (\exists y < x)(\mathcal{P}rf_{\beta}(\overline{\nu_{\beta}}, y)) \quad .$$

Then α' is a RE-formula in T because $\mathcal{P}rf_{\beta}(\overline{\nu_{\beta}}, y)$ is a PR-formula in T . For $m \in A$ we have $T \vdash \alpha(\overline{m})$ and hence $T \vdash \alpha'(\overline{m})$. If $m \notin A$ then $T \vdash \neg \alpha(\overline{m})$ $T \vdash \neg(\exists y < \overline{m})(\mathcal{P}rf_{\beta}(\overline{\nu_{\beta}}, y))$ where from we get $T \vdash \neg \alpha'(\overline{m})$. We have shown $\alpha \in \mathcal{B}in_T^A(RE)$. From the definition of α' we obtain $T \vdash \alpha(x) \longrightarrow \alpha'(x)$ which means $\alpha \leq_T \alpha'$. We know that $S \not\vdash \nu_{\beta}$ and hence

(1) $$T \not\vdash Con_{\alpha} \longrightarrow \nu_{\beta} \quad .$$

We show

(2) $$T \vdash Con_{\alpha'} \longrightarrow \nu_{\beta} \quad .$$

We have

$$T \vdash \neg \nu_{\beta} \longrightarrow (\exists y)(\mathcal{P}rf_{\beta}(\overline{\nu_{\beta}}, y)) \quad ,$$

$$T \vdash \qquad \longrightarrow (\exists y)(\forall x > y)(\alpha'(x) \equiv Fm^*(x)) \quad ,$$

$$T \vdash \qquad \longrightarrow \neg Con_{\alpha'} \quad .$$

If $\alpha' \leq_T \alpha$, i.e. if $T \vdash Con_{\alpha} \longrightarrow Con_{\alpha'}$ we obtain $T \vdash Con_{\alpha} \longrightarrow \nu_{\beta}$ by (2); but this contradicts (1).

We shall not prove in detail all statements of the paper [2] for the lattice of RE-bi-numerations, but we will show the method how to convert some proofs for $\mathcal{B}in_T^{A_0}(PR)$ (where A_0 is a primitive recursive axiomatization of T) to the proofs of analogous statements for $\mathcal{B}in_T^{A}(RE)$. Even if in premises of some theorems for the lattice of PR-bi-numerations the requirement of ω-consistency of T did not occur, in premises of analogous theorems for the lattice of RE-bi-numerations this presumption must be added.

Most of the proofs in [2] are performed constructions of the following type: For $\alpha \in \mathcal{B}in_T^{A_0}(PR)$ one constructs a formula $F(\alpha)$ which preserves the property "to be a PR-formula". Then we set $\alpha'(x) = \alpha(x) \,\&\, F(\alpha)(x)$ or $\alpha(x) = \alpha(x) \vee F(\alpha)(x)$. Clearly, α' is a PR-formula. The formula $F(\alpha)(x)$ is constructed in such a way that α' has required properties and $\alpha' \in \mathcal{B}in_T^{A_0}(PR)$.

The most fundamental properties of $F(\alpha)$ for the proof of the required properties of α' depend only on properties of the formula $\dot{\mathcal{P}r}_\alpha(x)$ and in fact that formula $\dot{\mathcal{P}rf}_\alpha(x, y)$ bi-numerates $\mathcal{P}rf_{A_0}(\varphi, d)$ in T . But this procedure often fails when applied to formulas from $\mathcal{B}in_T^{A}(RE)$. The main reason is that $\dot{F}$ need not save the property "to be an RE-formula".

This obstacle can be removed by the following procedure: For $\alpha \in \mathcal{B}in_T^{A}(RE)$, we can construct a primitive recursive axiomatization A_0 and its PR-bi-numeration α_0 in T such that $T \vdash \mathcal{P}r_{\alpha}(x) \equiv \mathcal{P}r_{\alpha_0}(x)$ by the construction described in Theorem 1.5. By our assumptions, T is ω-consistent. For this α_0 we construct $F(\alpha_0)$ according to the proof of the relevant statement for the lattice of PR-bi-numerations and finally we put $\alpha'(x) = \alpha(x) \,\&\, F(\alpha_0)(x)$ or $\alpha'(x) = \alpha(x) \vee F(\alpha_0)(x)$. Since α_0 is a PR-formula, $F(\alpha_0)$ is also a PR-formula. Now it is obvious that α' is an RE-formula in T . Since the fundamental properties of the formula $F(\alpha_0)$ depend on $\mathcal{P}r_{\alpha_0}(x)$ and α_0 was constructed so that $T \vdash \mathcal{P}r_{\alpha} \equiv \mathcal{P}r_{\alpha_0}$, we can prove that the formulas α and α' are related in the same way as the relevant formulas from $\mathcal{B}in_T^{A_0}(PR)$.

In this manner we can convert the proof of the required statement for the lattice of RE-bi-numerations into the proof of the analogous statement for the lattice of PR-bi-numerations. We can illustrate this procedure by the following figure:

$$\begin{tikzcd}
& F(\alpha) \arrow[dr, "\alpha' = \alpha \,\&\, F(\alpha)"] & \\
\alpha \arrow[ur] \arrow[rr] & & \alpha' = \alpha \vee F(\alpha)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \alpha_0 \arrow[r] & F(\alpha_0) \arrow[dr, "\alpha' = \alpha \,\&\, F(\alpha_0)"] & \\
\alpha \arrow[ur] \arrow[rrr] & & & \alpha' = \alpha \vee F(\alpha_0)
\end{tikzcd}$$

Construction in $\mathcal{B}in_T^{A_0}(PR)$ | Construction in $\mathcal{B}in_T^{A}(RE)$

Thus we can prove the following theorem (numbers of the corresponding statements from [2] for the lattice of PR-bi-numerations are in brackets):

2.2. Theorem. If T is a reflexive theory then in $\langle \mathcal{B}in_T^A(\mathcal{RE})\rangle$ there is no minimal element.

2.3. Theorem [2.11]. For each $\alpha, \beta \in \mathcal{B}in_T^A(\mathcal{RE})$, $\alpha \leq_T \beta$ iff there is a $\beta' \in \mathcal{B}in_T^A(\mathcal{RE})$ such that

1) $\beta =_T \beta'$,

2) $T \vdash \alpha(x) \longrightarrow \beta'(x)$.

2.4. Theorem [2.12]. For each $\alpha_1, \alpha_2 \in \mathcal{B}in_T^A(\mathcal{RE})$ if $\alpha_1 <_T \alpha_2$ then there is an $\alpha \in \mathcal{B}in_T^A(\mathcal{RE})$ such that $\alpha_1 <_T \alpha < \alpha_2$.

2.5. Theorem [2.14]. Let T be a reflexive theory. Then for each $\alpha \in \mathcal{B}in_T^A(\mathcal{RE})$ there is an $\alpha' \in \mathcal{B}in_T^A(\mathcal{RE})$ such that simultaneously $\alpha' \nleq_T \alpha$ and $\alpha \nleq_T \alpha'$.

2.6. Theorem [2.19], [2.21]. In $\langle \mathcal{B}in_T^A(\mathcal{RE})\rangle$ every pair $[\alpha]_{\langle \mathcal{B}in_T^A(\mathcal{RE})\rangle}$, $[\beta]_{\langle \mathcal{B}in_T^A(\mathcal{RE})\rangle}$ has the maximum and the infimum.

2.7. Corollary [2,20, 2.22]. Let $\alpha_1, \alpha_2, \alpha \in \mathcal{B}in_T^A(\mathcal{RE})$; then $[\alpha]_{\langle \mathcal{B}in_T^A(\mathcal{RE})\rangle}$ is the supremum and infimum of the pair $[\alpha_1]_{\langle \mathcal{B}in_T^A(\mathcal{RE})\rangle}$, $[\alpha_2]_{\langle \mathcal{B}in_T^A(\mathcal{RE})\rangle}$ respectively iff $T \vdash \dot{C}on_\alpha \equiv \dot{C}on_{\alpha_1} \vee \dot{C}on_{\alpha_2}$ and $T \vdash \dot{C}on_\alpha \equiv \dot{C}on_{\alpha_1} \,\&\, \dot{C}on_{\alpha_2}$ respectively.

This enables us to define on $\langle \mathcal{B}in_T^A(\mathcal{RE})\rangle$ the operations of join $\cup$ and meet $\cap$ similarly as in [2], 2.23.

2.8. Summary. From Corollary 2.7 it follows that $\langle \mathcal{B}in^{A}_{T}(RE)\rangle$ with operations $\cup$, $\cap$ is a distributive lattice which has no maximal element and if, in addition, T is reflexive, it has no minimal element.

A very important theorem of the paper [2] is Theorem 3.9 on Σ_1-nondefinability. The reader verifies easily that the whole proof of [2], 3.9 works also for $\langle \mathcal{B}in^{A}_{T}(RE)\rangle$ if modified according to our Figure. Thus we have the following

2.9. Theorem on Σ_1-non-definability [3.9]. Let T be reflexive. Then no k-tuple of elements of $\langle \mathcal{B}in^{A}_{T}(RE)\rangle$ is Σ_1-definable in $\langle \mathcal{B}in^{A}_{T}(RE)\rangle$.

§ 3. The lattices of numerations

In § 2 we have shown that $\langle \mathcal{B}in^{A}_{T}(RE)\rangle$ is a lattice with various interesting properties. In this section we shall study the relations between the structures

$\langle \mathcal{N}um^{A}_{T}(RE)\rangle$, $\langle \mathcal{B}in^{A}_{T}(RE)\rangle$, $\langle \mathcal{N}um^{\infty}_{T}(RE)\rangle$,
$\langle \mathcal{B}in^{\infty}_{T}(RE)\rangle$, $\langle \mathcal{B}in^{\infty}_{T}(PR)\rangle$, $\langle \mathcal{B}in^{A}_{T}(PR)\rangle$.

We shall show that all these structures are lattices and that they are mutually isomorphic except $\langle \mathcal{B}in^{A}_{T}(PR)\rangle$. In this section we shall assume that T is primitively recursively axiomatizable, ω-consistent and that $P \subseteq T$.

3.1. Lemma. The following equation holds:

$$\mathcal{B}in_T^{\infty}(PR) = \mathcal{N}um_T^{\infty}(PR) .$$

If A is a primitive recursive axiomatization of T then

$$\mathcal{B}in_T^{A}(PR) = \mathcal{N}um_T^{A}(PR) .$$

Proof: Since T is primitively recursively axiomatizable, the structures $\mathcal{B}in_T^{\infty}(PR)$, $\mathcal{N}um_T^{\infty}(PR)$ are not empty. T is a consistent theory and therefore by Lemma 1.3 we have $\mathcal{B}in_T^{A}(PR) \subseteq \mathcal{N}um_T^{A}(PR)$. Let α be a PR-numeration of an axiomatization A of T in T . We have $n \in A$ iff $T \vdash \alpha(\overline{n})$. Every PR-formula is a bi-numeration of a certain primitive recursive set $\overline{A}$ even in P and hence in T . Hence we have $n \in \overline{A} \Longrightarrow T \vdash \alpha(\overline{n})$, $n \notin \overline{A} \Longrightarrow T \vdash \neg\alpha(\overline{n})$. From the consistency of T it follows that $A = \overline{A}$.

Now we shall prove the fundamental statement for this section.

3.2. Theorem. Let A_2 be an arbitrary fixed recursively enumerable axiomatization of T . Then for every recursively enumerable axiomatization A_1 of T and for an arbitrary RE-numeration α_1 of A_1 in T we can construct an RE-numeration α_2 of A_2 such that the following holds:

$$(1) \qquad T \vdash Con_{\alpha_1} \equiv Con_{\alpha_2} .$$

If in addition $\mathcal{B}in_T^{A_2}(RE)$ is not empty (that is A_2 is recursive) then for every RE-numeration α_1 of A_1 in T we can construct an RE-bi-numeration α_2 of A_2 in T so that (1) holds.

Proof: Let α_{00} be an arbitrary RE-numeration (RE-bi-numeration if A_2 is recursive) of A_2 in T. We put $\alpha_0(x) = \alpha_{00}(x) \& \mathcal{P}r_{\alpha_1}(x)$. As $\alpha_{00}(x)$ and $\mathcal{P}r_{\alpha_1}(x)$ are RE-formulas in T, $\alpha_0(x)$ is also an RE-formula in T. We show that α_0 numerates (bi-numerates) A_2 in T.

Let $m \in A_2$. Then $T \vdash \alpha_{00}(\overline{m})$ and $T \vdash m$, hence $T \vdash \mathcal{P}r_{\alpha_1}(\overline{m})$, and consequently $T \vdash \alpha_{00}(\overline{m}) \& \mathcal{P}r_{\alpha_1}(\overline{m})$.

Let $m \notin A_2$. Then $T \nvdash \alpha_{00}(\overline{m})$ and hence $T \nvdash \alpha_{00}(\overline{m}) \& \& \mathcal{P}r_{\alpha_1}(\overline{m})$. If in addition α_{00} bi-numerates A_2 in T then $T \vdash \neg\, \alpha_{00}(\overline{m})$ and hence $T \vdash \neg\, \alpha_{00}(\overline{m}) \& \mathcal{P}r_{\alpha_1}(\overline{m})$.

The following sequence of statements is provable:

$$
\begin{aligned}
T &\vdash \neg\, \mathcal{C}on_{\alpha_0} && \equiv \mathcal{P}r_{\alpha_0}(\overline{0} \approx 1) \;, \\
T &\vdash && \equiv \mathcal{P}r_{\alpha_{00}} \& \mathcal{P}r_{\alpha_1}(\overline{0} \approx 1) \;, \\
T &\vdash && \longrightarrow \mathcal{P}r_{\mathcal{P}r_{\alpha_1}}(\overline{0} \approx 1) \;, \\
T &\vdash && \longrightarrow \mathcal{P}r_{\alpha_1}(\overline{0} \approx 1) \;, \\
T &\vdash && \longrightarrow \neg\, \mathcal{C}on_{\alpha_1} \;.
\end{aligned}
$$

From this we get

(1) $$T \vdash \mathcal{C}on_{\alpha_1} \longrightarrow \mathcal{C}on_{\alpha_0} \;.$$

According to Theorem 1.5, we construct a PR-formula α'_1 for the formula α_1 such that $T \vdash \mathcal{P}r_{\alpha_1} \equiv \mathcal{P}r_{\alpha'_1}$. Finally we put:

$$\alpha_2(x) = \alpha_0(x) \vee Fm^*(x) \& (\exists y < x)(\mathcal{P}rf_{\alpha'_1}(\overline{0 \approx 1}, y)) .$$

The following sequence of implications holds:

$T \vdash \neg Con_{\alpha_1} \to (\exists y)(Prf_{\alpha_1}(\overline{0 \approx 1}, y))$,

$T \vdash \qquad\qquad \to (\exists y)(Prf_{\alpha'_1}(\overline{0 \approx 1}, y))$,

$T \vdash \qquad\qquad \to (\exists y)(\forall x > y)(\exists z < x)(Prf_{\alpha'_1}(\overline{0 \approx 1}, z))$.

From this we get $T \vdash \neg Con_{\alpha_1} \to (\exists y)(\forall x > y)(\alpha_2(x) \equiv Fm^*(x))$.

It is easy to see that $T \vdash (\exists y)(\forall x > y)(\alpha_2(x) \equiv Fm^*(x)) \to$ $\to \neg Con_{\alpha_2}$ and hence $T \vdash \neg Con_{\alpha_1} \to \neg Con_{\alpha_2}$, which implies

$$(2) \qquad T \vdash Con_{\alpha_2} \to Con_{\alpha_1} .$$

We prove $T \vdash Con_{\alpha_1} \to Con_{\alpha_2}$.

It holds:

$T \vdash Con_{\alpha_1} \to (\neg \exists y)(Prf_{\alpha_1}(\overline{0 \approx 1}, y))$,

$T \vdash \qquad\qquad \to (\neg \exists y)(Prf_{\alpha'_1}(\overline{0 \approx 1}, y))$,

$T \vdash \qquad\qquad \to \alpha_2(x) \equiv \alpha_0(x)$,

$T \vdash \qquad\qquad \to (Con_{\alpha_0} \to Con_{\alpha_2})$.

Consequently, $T \vdash Con_{\alpha_1} \to (Con_{\alpha_0} \to Con_{\alpha_2})$, from which we obtain

$$T \vdash (Con_{\alpha_1} \to Con_{\alpha_0}) \to (Con_{\alpha_1} \to Con_{\alpha_2}) .$$

The last statement gives $T \vdash Con_{\alpha_1} \to Con_{\alpha_2}$ by (1).

Now it is necessary to prove that α_2 RE-numerates (RE-bi-numerates) A_2 in T. Clearly, α_2 is a RE-formula in T. Suppose $n \in A_2$. Then $T \vdash \alpha_0(\overline{n})$ because α_0 is numeration of A_2 in T and hence $T \vdash \alpha_0(\overline{n})$ by the construction of α_2 . Since $\mathcal{P}\!r\!f_{\alpha'_1}(x, y)$ is a PR-formula in T we have the following for each integer m : $T \vdash \mathcal{P}\!r\!f_{\alpha'_1}(\overline{0 \approx 1}, \overline{m})$ or $T \vdash \neg \mathcal{P}\!r\!f_{\alpha'_1}(\overline{0 \approx 1}, \overline{m})$. Since T is consistent, we have $T \not\vdash \mathcal{P}\!r\!f_{\alpha'_1}(\overline{0 \approx 1}, \overline{m})$ and hence $T \vdash \neg \mathcal{P}\!r\!f_{\alpha'_1}(\overline{0 \approx 1}, \overline{m})$ for each m . From this it follows that

(3) $$T \vdash \neg (\exists y < \overline{m})(\mathcal{P}\!r\!f_{\alpha'_1}(\overline{0 \approx 1}, y))$$

and by the consistency of T we have

(4) $$T \not\vdash (\exists y < \overline{m})(\mathcal{P}\!r\!f_{\alpha'_1}(\overline{0 \approx 1}, y))$$

for each integer m .

Suppose $n \notin A$; then $T \not\vdash \alpha_0(\overline{n})$ and by (4) we have $T \not\vdash \alpha_2(\overline{n})$. If in addition α_{00} was a bi-numeration of A in T, α_0 has also this property and $T \vdash \neg \alpha_0(\overline{n})$. By (3) we have $T \vdash \neg \alpha_2(\overline{n})$.

3.3. <u>Theorem</u>. Let A_1 , A_2 be recursively enumerable axiomatizations of T .

1) There exists an isomorphism of $\langle \mathcal{N}um_T^{A}(RE) \rangle$ and $\langle \mathcal{N}um_T^{A_2}(RE) \rangle$. We write $\langle \mathcal{N}um_T^{A_1}(RE) \rangle \approx \langle \mathcal{N}um_T^{A_2}(RE) \rangle$.

2) If in addition $\mathcal{B}in_T^{A_2}(RE) \neq 0$, i.e. if A_2 is recursive, then $\langle \mathcal{N}um_T^{A_1}(RE) \rangle \not\approx \langle \mathcal{B}in_T^{A_2}(RE) \rangle$.

3) If $\mathcal{B}in_T^{A_1}(RE) \neq 0$ and $\mathcal{B}in_T^{A_2}(RE) \neq 0$ then

$$\langle \mathcal{B}in_T^{A_1}(RE)\rangle \approx \langle \mathcal{B}in_T^{A_2}(RE)\rangle .$$

Proof: According to Theorem 3.2 for every $\alpha_1 \in \mathcal{N}um_T^{A_1}(RE)$ we can construct an $\alpha_2 \in \mathcal{N}um_T^{A_2}(RE)$ so that $T \vdash \mathcal{C}on_{\alpha_1} \equiv \mathcal{C}on_{\alpha_2}$. Denote by f the mapping which assigns the formula $f(\alpha_1)$ constructed in the proof of Theorem 3.2, for each formula α_1. We define a function $G : \langle \mathcal{N}um_T^{A_1}(RE)\rangle \longrightarrow \langle \mathcal{N}um_T^{A_2}(RE)\rangle$ in the following way: $G([\alpha_1]_{\langle \mathcal{N}um_T^{A_1}(RE)\rangle}) = [f(\alpha_1)]_{\langle \mathcal{N}um_T^{A_2}(RE)\rangle}$.
We must prove that G is correctly defined, i.e. that G is one-one, onto, and preserves the ordering $\leq_T$.

a) G is correctly defined. Let $[\alpha_1]_{\langle \mathcal{N}um_T^{A_1}(RE)\rangle} = [\alpha'_1]_{\langle \mathcal{N}um_T^{A_1}(RE)\rangle}$; then $T \vdash \mathcal{C}on_{\alpha_1} \equiv \mathcal{C}on_{\alpha'_1}$.
From the properties of f we obtain $T \vdash \mathcal{C}on_{\alpha_1} \equiv \mathcal{C}on_{f(\alpha_1)}$, $T \vdash \mathcal{C}on_{\alpha'_1} \equiv \mathcal{C}on_{f(\alpha'_1)}$ and hence $T \vdash \mathcal{C}on_{f(\alpha_1)} \equiv \mathcal{C}on_{f(\alpha'_1)}$, which implies

$$[f(\alpha_1)]_{\langle \mathcal{N}um_T^{A_2}(RE)\rangle} = [f(\alpha'_1)]_{\langle \mathcal{N}um_T^{A_2}(RE)\rangle} .$$

b) G preserves $\leq_T$. Let $[\alpha_1]_{\langle \mathcal{N}um_T^{A_1}(RE)\rangle} \leq_T$ $\leq_T [\beta_1]_{\langle \mathcal{N}um_T^{A_1}(RE)\rangle}$, i.e. $T \vdash \mathcal{C}on_{\beta_1} \longrightarrow \mathcal{C}on_{\alpha_1}$. Since $T \vdash \mathcal{C}on_{\beta_1} \equiv \mathcal{C}on_{f(\beta_1)}$ and $T \vdash \mathcal{C}on_{\alpha_1} \equiv \mathcal{C}on_{f(\alpha_1)}$ we can write $T \vdash \mathcal{C}on_{f(\beta_1)} \longrightarrow \mathcal{C}on_{f(\alpha_1)}$ which implies

$$G([\alpha_1]_{\langle \mathcal{N}um_T^{A_1}(RE)\rangle}) \leq_T G([\beta_1]_{\langle \mathcal{N}um_T^{A_1}(RE)\rangle}) .$$

c) G is onto. For the proof of this statement it is sufficient to show that for every $\alpha_2 \in \mathcal{N}um_T^{A_2}(RE)$ there exists an $\alpha_1 \in \mathcal{N}um_T^{A_1}(RE)$ so that $T \vdash Con_{\alpha_2} \equiv Con_{\alpha_1}$, which is guaranteed by Theorem 3.2.

d) G is one-one. Since $T \vdash Con_{\alpha_1} \equiv Con_{f(\alpha_1)}$ and $T \vdash Con_{\beta_1} \equiv Con_{f(\beta_1)}$, we have: $T \vdash Con_{f(\alpha_1)} \equiv Con_{f(\beta_1)}$ iff $T \vdash Con_{\alpha_1} \equiv Con_{\beta_1}$.

Analogously for 2, 3.

3.4. Theorem. Let A be a recursively enumerable axiomatization of T. Then

1) $\langle \mathcal{N}um_T^{\infty}(RE)\rangle \approx \langle \mathcal{N}um_T^{A}(RE)\rangle$,

2) $\langle \mathcal{B}in_T^{\infty}(RE)\rangle \approx \langle \mathcal{N}um_T^{A}(RE)\rangle$,

3) $\langle \mathcal{B}in_T^{A}(RE)\rangle \neq \emptyset \Rightarrow \langle \mathcal{N}um_T^{\infty}(RE)\rangle \approx \langle \mathcal{B}in_T^{A}(RE)\rangle$,

4) $\mathcal{B}in_T^{A}(RE) \neq \emptyset \Rightarrow \langle \mathcal{B}in_T^{\infty}(RE)\rangle \approx \langle \mathcal{B}in_T^{A}(RE)\rangle$.

Remark. Since T is primitive recursive axiomatizable, we have the following:

$\mathcal{N}um_T^{\infty}(RE) \neq \emptyset$, $\mathcal{B}in_T^{\infty}(RE) \neq \emptyset$, $\mathcal{B}in_T^{\infty}(PR) \neq \emptyset$.

Proof of Theorem 3.4: Let $\alpha \in \mathcal{N}um_T^{\infty}(RE)$. By Theorem 3.2 there exists a mapping $f: \mathcal{N}um_T^{\infty}(RE) \longrightarrow \mathcal{N}um_T^{A}(RE)$ so that for every $\alpha \in \mathcal{N}um_T^{\infty}(RE)$ we have $T \vdash Con_{\alpha} \equiv Con_{f(\alpha)}$. Define a mapping $H: \langle \mathcal{N}um_T^{\infty}(RE)\rangle \longrightarrow$ $\longrightarrow \langle \mathcal{N}um_T^{A}(RE)\rangle$ by the equation

$$H([\alpha]_{\langle \mathcal{N}um_T^{\infty}(RE)\rangle}) = [f(\alpha)]_{\langle \mathcal{N}um_T^{A}(RE)\rangle}.$$

Similarly as in Theorem 3.3 we can prove that G is correctly defined and is an isomorphism. Analogously for 2, 3, 4.

3.5. Theorem. 1) $\langle \mathcal{N}um_T^{\infty}(RE)\rangle \approx \langle \mathcal{B}in_T^{\infty}(PR)\rangle$,

2) $\langle \mathcal{B}in_T^{\infty}(RE)\rangle \approx \langle \mathcal{B}in_T^{\infty}(PR)\rangle$.

Proof: In the proof of Theorem 1.5, a formula $g(\alpha) \in \mathcal{B}in_T^{\infty}(PR)$ was constructed for every $\alpha \in \mathcal{N}um_T^{\infty}(RE)$ such that

(1) $$T \vdash \dot{\mathcal{P}r}_{\alpha}(x) \equiv \dot{\mathcal{P}r}_{g(\alpha)}(x) \quad .$$

From (1) we have

(2) $$T \vdash \dot{\mathcal{C}on}_{\alpha} \equiv \dot{\mathcal{C}on}_{g(\alpha)} \quad .$$

Define a function $K : \langle \mathcal{N}um_T^{\infty}(RE)\rangle \longrightarrow \langle \mathcal{B}in_T^{\infty}(PR)\rangle$ by the following equation:

$$K([\alpha]_{\langle \mathcal{N}um_T^{\infty}(RE)\rangle}) = [g(\alpha)]_{\langle \mathcal{B}in_T^{\infty}(PR)\rangle} \quad .$$

Similarly as in 3.3 we can prove that K is correctly defined, one-one and that it preserves the ordering $\preccurlyeq_T$.

We have to prove that K is onto. Suppose $[\beta]_{\langle \mathcal{B}in_T^{\infty}(PR)\rangle} \in \langle \mathcal{B}in_T^{\infty}(PR)\rangle$; then $[\beta]_{\langle \mathcal{N}um_T^{\infty}(RE)\rangle} \in \langle \mathcal{N}um_T^{\infty}(RE)\rangle$. Since $T \vdash \dot{\mathcal{C}on}_{\beta} \equiv \dot{\mathcal{C}on}_{g(\beta)}$, we have $[\beta]_{\langle \mathcal{B}in_T^{\infty}(PR)\rangle} = [g(\beta)]_{\langle \mathcal{B}in_T^{\infty}(PR)\rangle}$ and hence $K([\beta]_{\langle \mathcal{N}um_T^{\infty}(RE)\rangle}) = [\beta]_{\langle \mathcal{B}in_T^{\infty}(PR)\rangle}$.

3.6. Summary. Let A_1 , A_2 be arbitrary recursive enumerable axiomatizations of the theory T . Then the following holds:

$\langle \mathcal{N}um_T^{A_1}(RE)\rangle \approx \langle \mathcal{N}um_T^{A_2}(RE)\rangle \approx \langle \mathcal{N}um_T^{\infty}(RE)\rangle \approx$

$\approx \langle \mathcal{B}in_T^{\infty}(RE)\rangle \approx \langle \mathcal{B}in_T^{\infty}(PR)\rangle = \langle \mathcal{N}um_T^{\infty}(PR)\rangle$.

If in addition $\mathcal{B}in_T^{A_2}(RE) \neq 0$ (that is if A_2 is recursive) then $\langle \mathcal{B}in_T^{A_2}(RE)\rangle$ is isomorphic with all above mentioned structures.

3.7. Corollary. All above mentioned structures are lattices. Each of the above mentioned structures has the same properties as $\langle \mathcal{B}in_T^{A}(RE)\rangle$ which was studied in § 2.

An open problem: whether for a primitive recursive axiomatization A of the theory T one has $\langle \mathcal{B}in_T^{A}(PR)\rangle \approx$ $\approx \langle \mathcal{B}in_T^{\infty}(PR)\rangle \approx \ldots$ etc. For a proof of this statement it would be sufficieht to show that there exists a primitive recursive axiomatization A_{00} of T such that for every primitive recursive axiomatization A and for arbitrary PR-bi-numeration α of A in T there exists a PR-bi-numeration α_{00} of A_0 in T such that

$$T \vdash \dot{Con}_{\alpha} \longrightarrow \dot{Con}_{\alpha_{00}} \quad . \tag{5}$$

Now we could construct a PR-bi-numeration α_0 of A_{00} putting $\alpha_0 = \alpha_{00}(x) \vee Fm^*(x) \,\&\, (\exists y < x)(\dot{\mathcal{P}rf}_{\alpha}(\overline{0 \approx 1}, y))$ according to the second part of the proof of Theorem 3.2 for which we have: $T \vdash \dot{Con}_{\alpha} \equiv \dot{Con}_{\alpha_0}$. The construction of an isomorphism between $\langle \mathcal{B}in_T^{A_0}(PR)\rangle$ and $\langle \mathcal{B}in_T^{\infty}(PR)\rangle$ should be similar as the construction of the function $\mathcal{H}$ in Theorem 3.3. However, I have not succeeded to prove or disprove the existence of α_{00} . To close, let us mention that if we succeed to prove the existence of an isomorphism between $\langle \mathcal{B}in_T^A(PR) \rangle$ and $\langle \mathcal{B}in_T^{\infty}(PR) \rangle$, all studied structures shall have the same properties as lattices. In this case the procedure for converting proofs for $\langle \mathcal{B}in_T^{A_0}(PR) \rangle$ to relevant proofs for $\langle \mathcal{B}in_T^A(RE) \rangle$ will lose its importance.

R e f e r e n c e s

[1] FEFERMAN S.: Arithmetization of metamathematics in a general setting, Fund.Math.XLIX(1960),36-92.

[2] HÁJKOVÁ M.: The lattice of bi-numerations of arithmetic, Comment.Math.Univ.Carolinae 12(1971),81-104,281-306.

[3] MENDELSON E.: Introduction to mathematical logic, Moskva,1971.

[4] TARSKI A., MOSTOWSKI A., ROBINSON R.M.: Undecidable theories, Amsterdam,1953.

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(Oblatum 19.4.1973)