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CARDINAL SUMS AND DIRECT PRODUCTS IN GALOIS CONNECTIONS

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Abstract: In this paper we study Galois connections, especially the Galois connections of cardinally and directly decomposable posets. We describe the poset of all polarized mappings of cardinally decomposable posets. We give a characterization of polarized mappings and study three problems of [3]; we show by examples that, in some cases, they have negative answers. We give a partial solution for one of these problems concerning posets of all polarized mappings between two posets, where one of these posets is directly decomposable.

Key words and phrases: Galois connections, polarized mappings, cardinal sums and direct products of posets, cardinally indecomposable posets.

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1. Introduction. In this paper we study Galois connections, especially the Galois connections of cardinally and directly decomposable posets.

Let A, B be posets, $T: A \to B$, $T^*: B \to A$ be antitone mappings with $T^*T(a) \geq a$, $TT^*(b) \geq b$ for all $a \in A, b \in B$; then $\langle T, T^* \rangle$ is called a Galois connection between A, B; T is a polarized mapping.

The poset of all polarized mappings $A \to B$ will be denoted by $\mathcal{L}(A, B)$ (cf. [2],[3]).

We use these symbols and assumptions:

$\times$ (resp. $\oplus$) signifies the symbol for the direct product (resp. for the cardinal sum);

$\mathbb{1}$ denotes a singleton, $\mathbb{2}$ denotes a chain consisting of two elements; the empty set $\emptyset$ is a poset as well as a lattice.

Let $\langle T, T^* \rangle$ be a Galois connection between posets A, B let $a \in A$, $b \in B$. Then, as it is well known,

(1) T^* is uniquely defined by T;

(2) $a \leq T^*(b)$ iff $b \leq T(a)$.

In [3], among others, these problems are formulated:

3a) A, B, C be given posets. Is
$\mathcal{L}(A, \mathcal{L}(B, C)) \cong \mathcal{L}(\mathcal{L}(A, B), C)$?

3b) A, B, C be given posets. Is
$\mathcal{L}(A, B \times C) \cong \mathcal{L}(A, B) \times \mathcal{L}(A, C)$?

5) Does $\mathcal{L}(A, A) \cong \mathcal{L}(B, B)$ imply that $A \cong B$?

Theorem 9 represents a partial solution of the problem 3b). Generally, there is no positive solution for these problems, which is demonstrated by examples in the last section.

2. Cardinal sums. It is well known that it is possible to describe every poset as a cardinal sum of cardinally indecomposable posets; we get this cardinal sum as a decomposition of this poset according to the equivalence relation generated by the relation $\not\parallel$.

If A or B is empty, then evidently $\mathcal{L}(A,B)=\emptyset$. We shall study cardinally decomposable nonempty posets.

<u>Theorem 1</u>. Let A, B be posets, $A=\bigoplus_{i\in I}A_i$, $B=\bigoplus_{j\in J}B_j$, where $A_i\neq\emptyset$, $B_j\neq\emptyset$ are cardinally indecomposable posets for all $i\in I$, $j\in J$. Then $\mathcal{L}(A,B)$ is not empty iff there exists a mapping $\varphi: I\longrightarrow J$ such that φ is onto, one-to-one and for all $i\in I$, $\mathcal{L}(A_i,B_{\varphi(i)})$ is not empty.

<u>Proof</u>. 1) Let $T\in\mathcal{L}(A,B)$, let $i\in I$, $a_i\in A_i$ be arbitrary. If $T(a_i)\in B_j$, we denote $\varphi(i)=j$. Then $T(A_i)\subseteq B_{\varphi(i)}$ holds since any A_i is an equivalence class of A/ε where ε is the transitive closure for the relation $\mathcal{K}$. Let $a'_i\in A_i$, then $a_i\ \varepsilon\ a'_i$ and thus $T(a_i)\ \varepsilon\ T(a'_i)$. So, we have a mapping $\varphi: I\longrightarrow J$ which is one-to-one; φ is even onto. Indeed, if $j\in J$ is arbitrary, we take some $b_j\in B_j$. There exists $i\in I$ such that $T^*(b_j)\in A_i$. Since $TT^*(b_j)\geq b_j$, $TT^*(b_j)$ is contained in B_j and so $j=\varphi(i)$. Denote $T_i=T/_{A_i}$, $T_i^*=T^*/_{B_{\varphi(i)}}$. Obviously, $T_i\in\mathcal{L}(A_i,B_{\varphi(i)})$, $\langle T_i,T_i^*\rangle$ is a Galois connection between A_i, $B_{\varphi(i)}$.

2) Let φ be some onto and one-to-one mapping $I\longrightarrow J$, let there exist $T_i\in\mathcal{L}(A_i,B_{\varphi(i)})$ such that $\langle T_i,T_i^*\rangle$ is a Galois connection between A_i, $B_{\varphi(i)}$ for all $i\in I$. Define the mappings $T: A\longrightarrow B$, $T^*: B\longrightarrow A$ by $T(a)=T_i(a)$ if $a\in A_i$, $T^*(b)=T_i^*(b)$ if $b\in B_{\varphi(i)}$. Evidently, $\langle T,T^*\rangle$ forms a Galois connection between A, B.

Theorem 2. Let A, B be posets, $A = \bigoplus_{i \in I} A_i$, $B = \bigoplus_{j \in J} B_j$, where $A_i \neq \emptyset$, $B_j \neq \emptyset$ are cardinally indecomposable posets for all $i \in I$, $j \in J$. We denote by F the set of all onto and one-to-one mappings $I \longrightarrow J$. Then

$$\mathcal{L}(A,B) \cong \bigoplus_{\varphi \in F} \left(\times_{i \in I} \mathcal{L}(A_i, B_{\varphi(i)}) \right) .$$

Proof. 1) Obviously, in case F is empty, the assertion is true. Let F be nonempty. From the proof of the previous theorem we know that for every $T \in \mathcal{L}(A,B)$ there exists a unique mapping $\varphi_T \in F$ such that $\{T_i\}_{i \in I} \in \times_{i \in I} \mathcal{L}(A_i, B_{\varphi_T(i)})$ where $T_i = T/A_i$. Denote $\sigma(T) = \{T_i\}_{i \in I}$, i.e., $\sigma(T) \in \bigoplus_{\varphi \in F} (\times_{i \in I} \mathcal{L}(A_i, B_{\varphi(i)}))$.

2) Let $\{T_i\}_{i \in I} \in \bigoplus_{\varphi \in F} (\times_{i \in I} \mathcal{L}(A_i, B_{\varphi(i)}))$, i.e., there exists a mapping $\varphi \in F$ such that $\langle T_i, T_i^* \rangle$ is a Galois connection between A_i, $B_{\varphi(i)}$ for all $i \in I$. Denote $\eta(\{T_i\}_{i \in I}) = T$, where $T(a) = T_i(a)$ if $a \in A_i$, $T^*(b) = T_i^*(b)$ if $b \in B_{\varphi(i)}$. Then $\langle T, T^* \rangle$ forms a Galois connection between A, B.

Evidently:

a) If $T, G \in \mathcal{L}(A, B)$, $T \leq G$, then $\varphi_T = \varphi_G$ and $\sigma(T) \leq \sigma(G)$.

b) If $\{T_i\}_{i \in I}, \{G_i\}_{i \in I} \in \bigoplus_{\varphi \in F} (\times_{i \in I} \mathcal{L}(A_i, B_{\varphi(i)}))$, $\{T_i\}_{i \in I} \leq \{G_i\}_{i \in I}$, then $\eta(\{T_i\}_{i \in I}) \leq \eta(\{G_i\}_{i \in I})$.

c) $\sigma\eta(\{T_i\}_{i \in I}) = \{T_i\}_{i \in I}$, $\eta\sigma(T) = T$ for all $T \in \mathcal{L}(A, B)$, $\{T_i\}_{i \in I} \in \bigoplus_{\varphi \in F} (\times_{i \in I} \mathcal{L}(A_i, B_{\varphi(i)}))$.

3. Direct products. First we describe all polarized mappings between the posets A, B.

Theorem 3. Let A, B be posets, let $T: A \to B$ be an antitone mapping. Then $T \in \mathcal{L}(A, B)$ iff for every $b \in B$ there exists an element $a_b \in A$ such that $T(a_b) \geq b$; if $T(x) \geq b$ then $x \leq a_b$. Such element a_b is defined uniquely for every element b.

Proof. The uniqueness of a_b is obvious if a_b exists.

I) Let $T \in \mathcal{L}(A, B)$, denote $a_b = T^*(b)$. Certainly, $TT^*(b) = T(a_b) \geq b$; if $T(x) \geq b$, then $x \leq T^*(b) = a_b$.

II) Denote $T^*(b) = a_b$. In accordance with the uniqueness of a_b we get a mapping $T^*: B \to A$. $\langle T, T^* \rangle$ is a Galois connection between A, B. Indeed:

If $b \leq b'$, then $b \leq T(a_{b'})$ and thus $a_{b'} = T^*(b') \leq a_b = T^*(b)$.

If $b \in B$, then clearly $b \leq T(a_b) = TT^*(b)$,

if $a \in A$, then $a \leq a_{T(a)} = TT^*(a)$ since from $T(a) \leq T(a)$ we get $a \leq a_{T(a)}$.

Corollary 4. Let A, B be complete lattices, $T: A \to B$ be a mapping. Then $T \in \mathcal{L}(A, B)$ iff $T(0_A) = 1_B$ and $T(\bigcup_\alpha a_\alpha) = \bigcap_\alpha T(a_\alpha)$.

Proof. I) Let $T \in \mathcal{L}(A, B)$, then necessarily $T(0_A) = 1_B$. $T(\bigcup_\alpha a_\alpha) \leq T(a_\alpha)$ holds for any α; if $y \in B$, $y \leq T(a_\alpha)$ for any α, then $a_\alpha \leq T^*(y)$ for any α, thus $\bigcup_\alpha a_\alpha \leq T^*(y)$ and $y \leq T(\bigcup a_\alpha)$, i.e., $T(\bigcup_\alpha a_\alpha) = \bigcap_\alpha T(a_\alpha)$.

II) Let b be an arbitrary element of B , denote $a_b = \bigcup_{T(x) \geq b} x$. Then

$T(a_b) = T(\bigcup_{T(x) \geq b} x) = \bigcap_{T(x) \geq b} T(x) \geq b$,

$x \leq a_b$ if $T(x) \geq b$.

<u>Definition 5</u>. Let A, B, C be posets. A is said to satisfy the $\mathcal{L}$-condition with respect to B, C for $T_1 \in \mathcal{L}(A, B)$, $T_2 \in \mathcal{L}(A, C)$, if for any $b \in B$, $c \in C$ there exists $T_1^*(b) \cap T_2^*(c)$ in A . A satisfies the $\mathcal{L}$-condition with respect to B , C , if A satisfies the $\mathcal{L}$-condition with respect to B , C for any $T_1 \in \mathcal{L}(A, B)$, $T_2 \in \mathcal{L}(A, C)$.

E.g., every $\cap$-semilattice satisfies the $\mathcal{L}$-condition with respect to any posets B, C .

<u>Lemma 6</u>. Let A, B, C be posets, $T_1 \in \mathcal{L}(A, B)$, $T_2 \in \mathcal{L}(A, C)$. Define $T : A \longrightarrow B \times C$ as $T(a) = (T_1(a), T_2(a))$ for any $a \in A$. Then $T \in \mathcal{L}(A, B \times C)$ iff A satisfies the $\mathcal{L}$-condition with respect to B, C for T_1, T_2 .

<u>Proof</u>. I) Let A satisfy the $\mathcal{L}$-condition with respect to B, C for T_1, T_2 , define a mapping $T^* : B \times C \longrightarrow A$, $T^*(b, c) = T_1^*(b) \cap T_2^*(c)$. Then $\langle T, T^* \rangle$ is a Galois connection between A , $B \times C$ since
1) both T and T^* are antitone,

2) $TT^*(b, c) = (T_1(T_1^*(b) \cap T_2^*(c)), T_2(T_1^*(b) \cap T_2^*(c))) \geq$
$\geq (T_1T_1^*(b), T_2T_2^*(c)) \geq (b, c)$,
$T^*T(a) = T^*(T_1(a), T_2(a)) = T_1^*T_1(a) \cap T_2^*T_2(a) \geq a$.

II) Let $T_1 \in \mathcal{L}(A, B)$, $T_2 \in \mathcal{L}(A, C)$ and $T \in \mathcal{L}(A, B \times C)$, where $T(a) = (T_1(a), T_2(a))$. In accordance with Theorem 3, for any $(b, c) \in B \times C$ there exists an element $a_{bc} \in A$ such that $T(a_{bc}) \geq (b, c)$; $x \leq a_{bc}$ if $T(x) \geq (b, c)$. In particular, $T_1(a_{bc}) \geq b$, $T_2(a_{bc}) \geq c$, but thus $a_{bc} \leq a_b = T_1^*(b)$, $a_{bc} \leq a_c = T_2^*(c)$. Let $x \leq a_b, a_c$, then $T(x) = (T_1(x), T_2(x)) \geq (T_1T_1^*(b), T_2T_2^*(c)) \geq (b, c)$, i.e., $x \leq a_{bc}$ and $T_1^*(b) \cap T_2^*(c) = a_{bc} = T^*(b, c)$ really holds.

The following assertion is an immediate consequence of Lemma 6:

<u>Corollary 7</u>. If A, B, C are posets, then A satisfies the $\mathcal{L}$-condition with respect to B, C iff the poset $\mathcal{L}(A, B) \times \mathcal{L}(A, C)$ is so embeddable in $\mathcal{L}(A, B \times C)$ that for any $(T_1, T_2) \in \mathcal{L}(A, B) \times \mathcal{L}(A, C)$ there exists $T \in \mathcal{L}(A, B \times C)$ with $T(a) = (T_1(a), T_2(a))$ for all $a \in A$.

<u>Lemma 8</u>. Let A, B, C be posets, $T \in \mathcal{L}(A, B \times C)$. Define $T_1 : A \longrightarrow B$, $T_2 : A \longrightarrow C$ in this way: $T(a) = (T_1(a), T_2(a))$ for any $a \in A$. Then T_1, T_2 are polarized mappings iff for every $b \in B$ there exists some $c_b \in C$ such that for any $c \in C$, $TT^*(b, c) \geq (b, c_b)$; for every $c \in C$ there exists some $b_c \in B$ such that for any $b \in B$, $TT^*(b, c) \geq (b_c, c)$.

Proof. Let T_1, T_2 be polarized mappings. Then by Lemma 6 and its proof there is $T^*(b, c) = T_1^*(b) \cap T_2^*(c)$, i.e., $T^*(b, c) \leq T_1^*(b)$. Denote $c_b = T_2 T_1^*(b)$. We get $(b, c_b) \leq T T_1^*(b) = (T_1 T_1^*(b), T_2 T_1^*(b)) \leq T T^*(b, c)$ for any $c \in C$.

If we put $b_c = T_1 T_2^*(c)$, we can complete this part of the proof by a similar reasoning.

II) Let $\forall b \in B \ \exists c_b \in C \ \forall c \in C \ \ TT^*(b, c) \geq (b, c_b)$,

$$\forall c \in C \ \exists b_c \in B \ \forall b \in B \ \ TT^*(b, c) \geq (b_c, c) .$$

Denote $a_b = T^*(b, c_b)$, $a_c = T^*(b_c, c)$ for every $b \in B$, $c \in C$. The element a_b satisfies the conditions from Theorem 3 for T_1. Indeed, $T(a_b) = TT^*(b, c_b) \geq (b, c_b)$, thus $T_1(a_b) \geq b$; let $b \leq T_1(x)$, i.e., $T(x) = (T_1(x), T_2(x)) \geq (b, T_2(x))$, then, however, $T(x) = TT^*T(x) \geq TT^*(b, T_2(x)) \geq (b, c_b)$ and so $x \leq T^*(b, c_b) = a_b$. Analogous considerations hold for a_c and T_2 as well.

Theorem 9. Let A, B, C be posets. Then $\mathcal{L}(A, B) \times \mathcal{L}(A, C) \cong \mathcal{L}(A, B \times C)$ where the mappings $T \in \mathcal{L}(A, B \times C)$ and $(T_1, T_2) \in \mathcal{L}(A, B) \times \mathcal{L}(A, C)$ are in the correspondence when $T(a) = (T_1(a), T_2(a))$ holds for each $a \in A$, iff A satisfies the $\mathcal{L}$-condition with respect to B, C and for any $T \in \mathcal{L}(A, B \times C)$

$$\forall b \in B \ \exists c_b \in C \ \forall c \in C \ \ TT^*(b, c) \geq (b, c_b) ,$$

$$\forall c \in C \ \exists b_c \in B \ \forall b \in B \ \ TT^*(b, c) \geq (b_c, c) .$$

Proof. Theorem 9 is a consequence of Lemmas 6 and 8 and of Corollary 7.

Corollary 10. Let A, B, C be posets. Suppose that either $\mathcal{L}(A,B) = \mathcal{L}(A,C) = \mathcal{L}(A, B \times C) = \emptyset$ or that A satisfies the $\mathcal{L}$-condition with respect to B, C and that there exists $T \in \mathcal{L}(A, B \times C)$ (or resp. $T_1 \in \mathcal{L}(A,B)$ or resp. $T_2 \in \mathcal{L}(A,C)$) such that $T(A)$ has a lower bound in $B \times C$ (or resp. $T_1(A)$ in B or resp. $T_2(A)$ in C). Then $\mathcal{L}(A, B \times C) \cong \mathcal{L}(A,B) \times \mathcal{L}(A,C)$ and in the second case $1_A \in A$.

Proof. In the case $\mathcal{L}(A,B) = \mathcal{L}(A,C) = \mathcal{L}(A,B \times C) = \emptyset$ obviously $\mathcal{L}(A, B \times C) \cong \mathcal{L}(A,B) \times \mathcal{L}(A,C)$.
Let $T_1 \in \mathcal{L}(A,B)$, let $T_1(a) \geq b$ for each $a \in A$. Then $a \leq T_1^*(b)$ for each $a \in A$ which implies $T_1^*(b) = 1_A \in A$.
Let $T \in \mathcal{L}(A, B \times C)$ be arbitrary. Then $TT^*(b,c) \geq (b, c_1)$, $TT^*(b,c) \geq (b_1, c)$ for any $(b,c) \in B \times C$ where $(b_1, c_1) = T(1_A)$. By Theorem 9, the proof is complete.

Corollary 11. Let A, B, C be posets, let A satisfy the $\mathcal{L}$-condition with respect to B, C, let either $1_A \in A$ or $0_B \in B$ and $0_C \in C$. Then $\mathcal{L}(A, B \times C) \cong \mathcal{L}(A,B) \times \mathcal{L}(A,C)$.

Proof. By Theorem 9, this assertion is true.

4. Examples

Lemma 12. Let A, B be posets, let $T \in \mathcal{L}(A,B)$, a, b, a_1, $a_2 \in A$. Then the following assertions are true:

1) If $a \parallel b$, then either $T(a) \parallel T(b)$ or there

exists $c \in A$ such that $a \leq c$, $b \leq c$.

2) If a is a maximal element, then $T^*T(a) = a$ and $T(a)$ is a minimal element in $T(A)$.

3) If a_1, a_2 are different maximal elements, then $T(a_1) \parallel T(a_2)$.

<u>Proof</u>. 1) Let $T(a) \leq T(b)$, then $c = T^*T(a) \geq$ $\geq T^*T(b)$, i.e., $a \leq c$, $b \leq c$.

2) Necessarily $a \leq T^*T(a)$. In case a is maximal we get $a = T^*T(a)$; if $T(x) \leq T(a)$ then $a = T^*T(a) =$ $= T^*T(x)$ and thus $T(a) = T(x)$.

3) The remaining assertion is an obvious consequence of the previous ones.

<u>Example 13</u>. The poset A constructed below does not satisfy the $\mathcal{L}$-condition with respect to the posets B , C ; $\mathcal{L}(A, B \times C) \cong \mathcal{L}(A,B) \times \mathcal{L}(A,C)$ is not true, though $1_A \in A$, $0_B \in B$, $0_C \in C$ (cf. Fig. 1).

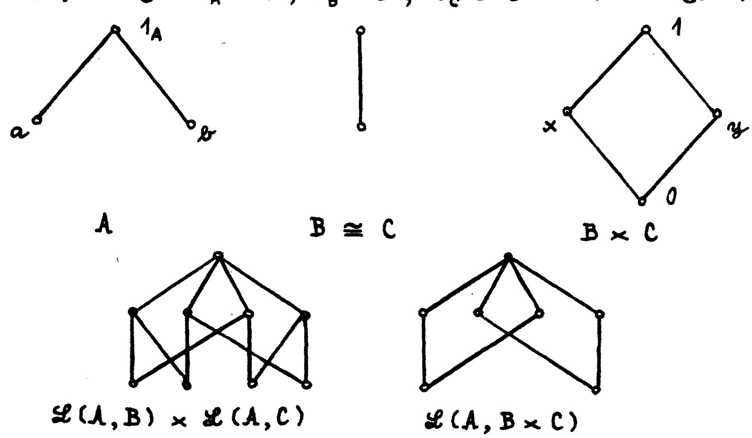

Fig. 1

$\mathcal{L}(A,B) \cong \mathcal{L}(A,C) \cong A$, cf.[3], Theorem 1.7. All polarized mappings from $\mathcal{L}(A, B \times C)$ are:

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
a	1	1	x	1	y	1	0
b	1	x	1	y	1	0	1
1_A	1	x	x	y	y	0	0

Tab. 1

By constructing $\mathcal{L}(A, B \times C)$ it is easily to obtain $\mathit{Card}\ T(A) \leqq 2$ and $1 \in T(A)$ if $T \in \mathcal{L}(A, B \times C)$.

Example 14. The poset A satisfies the $\mathcal{L}$-condition with respect to the posets B , C , but $1_A \notin A$, $0_B \notin B$, $0_C \notin C$. It follows that $\mathcal{L}(A, B \times C) \cong \mathcal{L}(A,B) \times \mathcal{L}(A,C)$ does not hold - cf. Fig. 2

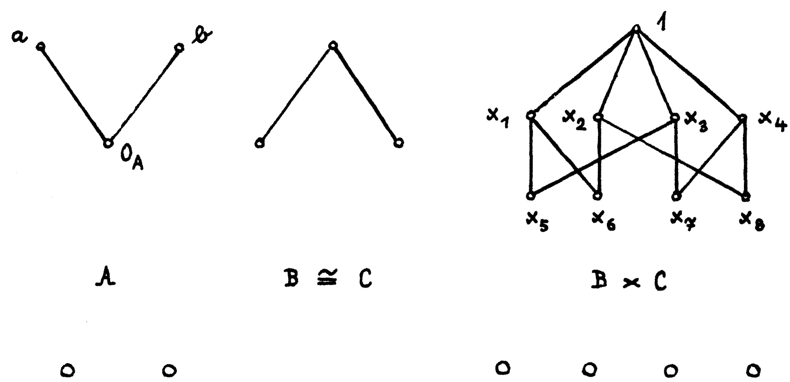

Fig. 2

There is no difficulty in proving: $\mathcal{L}(A,B) \times \mathcal{L}(A, C)$

is totally disordered (in the sense of [3]) since the posets $\mathcal{L}(A,B)$, $\mathcal{L}(A,C)$ are totally disordered, but $\mathcal{L}(A, B \times C)$ is not totally disordered: The mappings T, $T' \in \mathcal{L}(A, B \times C)$ defined by $T(a) = x_1$, $T(b) = x_4$, $T(0_A) = 1$; $T'(a) = x_5$, $T'(b) = x_8$, $T'(0_A) = 1$ are such that $T' \leq T$.

Example 15. Generally, it is not true that $\mathcal{L}(A, \mathcal{L}(B,C)) \cong \mathcal{L}(\mathcal{L}(A,B), C)$ for arbitrary posets A, B, C (cf. Fig. 3):

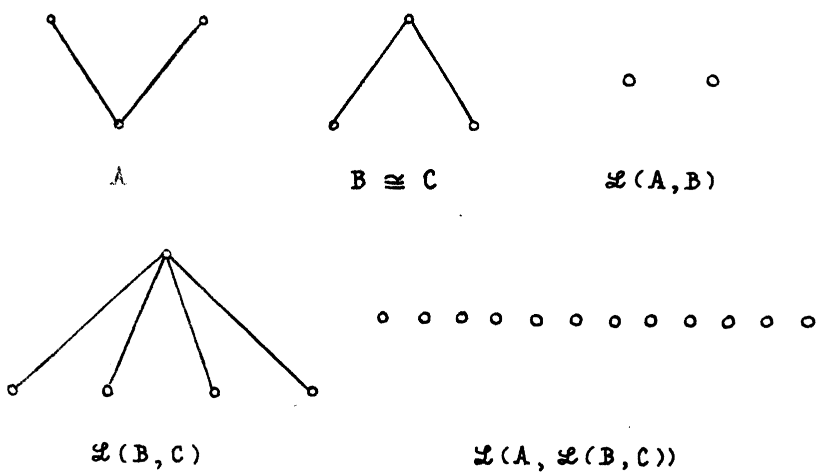

Fig. 3

$\mathcal{L}(A,B)$ is totally disordered but C is not totally disordered - thus in accordance with Theorem 1 $\mathcal{L}(\mathcal{L}(A,B), C)$ is empty; $\mathcal{L}(A, \mathcal{L}(B,C))$ is not empty.

Example 16. Generally, also the following conjecture does not hold:

"If A, B are posets, $\mathcal{L}(A,A) \cong \mathcal{L}(B,B)$, then $A \cong B$ ".

Indeed, for the posets A , B in Fig.4 we have

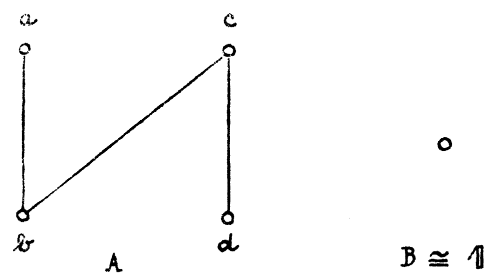

Fig.4

$\mathcal{L}(A,A) \cong \mathbb{1}$ since $T(a) \parallel T(c)$; $T(b) \geq T(a), T(c)$; $T(d) \geq T(c)$; $T^*T(a) = a$; $T^*T(c) = c$ by Lemma 12 and by definition of Galois connections, where $T \in \mathcal{L}(A,A)$. Clearly, $\mathcal{L}(B,B) \cong \mathbb{1}$, but $A \cong B$ does not hold.

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