Zdeněk Frolík; Jan K. Pachl

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PURE MEASURES

Zdeněk FROLÍK, Jan PACHL, Praha

Abstract: Pure measures (introduced by M.M. Rao [5]), and related classes of $\aleph_0$ -compact (E. Marczewski [2]) and purely $\aleph_0$ -compact (introduced below) measures are studied. All properties are equivalent for countably generated measures, every pure measure is perfect, and any indirect product of pure measures is a pure measure. Most of the natural questions are open.

Key words: Compact measure, perfect measure, pure measure, purely compact measure, indirect product of measures, Stone space.

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1. Definitions and notations.

1.1. Definition. (a) $\langle X, \mathcal{A}, \mu \rangle$ is a measure space if X is a non-empty set, $\mathcal{A} \subset \exp X$ is a σ -algebra and μ is a (positive finite σ -additive) measure on $\mathcal{A}$.

(b) Given a measure space $\langle X, \mathcal{A}, \mu \rangle$, $Y \in \mathcal{A}$ and $\mathcal{B} \subset \subset \mathcal{A}$ then $\mathcal{B}/Y = \{ E \cap Y \mid E \in \mathcal{B} \}$. μ/Y is the restriction of μ to $\mathcal{A}/Y$.

(c) A measure space $\langle X, \mathcal{A}, \mu \rangle$ (and measure μ) is countably-generated if there exists a countable algebra $\mathcal{A}_0 \subset \mathcal{A}$ such that $\mu E = \inf \{ \sum_{m=1}^{\infty} \mu B_m \mid B_m \in \mathcal{A}_0$ & & $\bigcup_{m=1}^{\infty} B_m \supset E \}$ for any $E \in \mathcal{A}$.

1.2. Definition ([2]). (a) A class $\mathcal{C} \subset \exp X$ is $\aleph_0$ -compact if for any countable $\mathcal{C}_0 \subset \mathcal{C}$ with $\bigcap \mathcal{C}_0 = \emptyset$ there is a finite $\mathcal{F} \subset \mathcal{C}_0$ with $\bigcap \mathcal{F} = \emptyset$.

(b) A measure μ on $\mathcal{A}$ is $\aleph_0$ -compact if there is an $\aleph_0$ -compact class $\mathcal{C} \subset \mathcal{A}$ such that

$\mu E = \sup \{\mu C \mid C \in \mathcal{C} \ \& \ C \subset E\}$ for any $E \in \mathcal{A}$.

1.3. Definition ([5]). (a) Given a measure space $\langle X, \mathcal{A}, \mu \rangle$ then a ring $\mathcal{R} \subset \mathcal{A}$ is μ -pure if

(i) $\mu E = \inf \{\sum_{n=1}^{\infty} \mu B_n / B_n \in \mathcal{R}, \bigcup B_n \supset E\}$ for any $E \in \mathcal{A}$

and (ii) $B_n \in \mathcal{R}$ for $n = 1, 2, \dots$, $B_n \searrow \emptyset$ imply $\mu B_k = 0$ for some k .

(b) Measure μ is pure if there exists a μ -pure algebra.

1.4. Remarks. (a) It suffices to suppose the existence of a μ -pure ring (instead of algebra) in 1.3 (b), see Proposition 2.5.

(b) There is a measure that is not pure (see [4] or [3] or 3.2 and [1], 49.3).

2. Basic Properties

2.1. Lemma. (a) Any strictly positive measure μ (i.e. $\mu E > 0$ for $E \in \mathcal{A}$, $E \neq \emptyset$) is pure.

(b) Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space, $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, $X_1, X_2 \in \mathcal{A}$; let μ / X_1 and μ / X_2 be pure. Then μ is pure.

(c) Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space. If a ring $\mathcal{R} \subset \mathcal{A}$ is μ -pure and $B \in \mathcal{R}$ then the algebra $\mathcal{R}/B$ is (μ / B) -pure.

Proof. (a) It suffices to consider a measure space $\langle N, \exp N, \mu \rangle$ where $N = \{1, 2, \dots\}$. Put $\mathcal{D} = \{E \subset N \mid E$ is finite and $1 \notin E\}$; then the algebra $\mathcal{D} \cup \mathcal{D}^c$ is μ-pure.

(b) Let $\mathcal{B}_i$ be (μ / X_i)-pure algebras, $i = 1, 2$. Then the algebra $\{E_1 \cup E_2 \mid E_i \in \mathcal{B}_i$ for $i = 1, 2\}$ is μ-pure.

(c) Obvious.

2.2. Lemma. Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space, $X = \bigcup_{n=0}^{\infty} X_n$ where $X_n \in \mathcal{A}$ are mutually disjoint; let $\mu(X_0) = 0$ and $X_0 \neq \emptyset$.

If μ / X_n are pure measures for $n = 1, 2, \dots$ then μ is a pure measure.

Proof. There exist (μ / X_n)-pure algebras $\mathcal{B}_n$. Put $\mathcal{B} = \{E \in \mathcal{A} \mid X_0 \subset E$ & there is an k such that $X_n \cap E \in \mathcal{B}_n$ for $1 \leq n \leq k$ and $X_n \subset E$ for $n > k\}$ then the algebra $\mathcal{B} \cup \mathcal{B}^c$ is μ-pure.

2.3. Proposition. Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space, $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathcal{A}$ are mutually disjoint. If μ / X_n are pure measures for $n = 1, 2, \dots$ then μ is a pure measure.

Proof. There exist (μ / X_n)-pure algebras $\mathcal{B}_n$. One may and shall assume that there is an E in $\mathcal{A}$ such that $\mu E = 0$ and $E \neq \emptyset$ (this follows from 2.1 (a)); then $E \cap X_k \neq \emptyset$ for some k. Pick up $p \in E \cap X_k$. Since Condition (i) in 1.3 (a) holds (for $\mathcal{R} = \mathcal{B}_k$) there exist $B_n \in \mathcal{B}_k$ for $n = 1, 2, \dots$ such that $p \in B_n$ and $\mu B_n < \frac{1}{n}$. Put

$$Y_0 = X_k \setminus B_1 ,$$
$$Y_n = \bigcap_{i=1}^{n} B_i \setminus B_{n+1} \qquad \text{for} \quad n = 1, 2, \dots$$
$$X_0 = \bigcap_{i=1}^{\infty} B_i \quad ;$$

Obviously $\mu X_0 = 0$, $X_0 \neq \emptyset$, and all the measures μ / Y_n ($n = 0, 1, 2, \dots$) are pure by 2.1 (c). Hence Lemma 2.2. applies to $X = \bigcup_{\substack{n=0 \\ n \neq k}}^{\infty} X_n \cup \bigcup_{n=0}^{\infty} Y_n$.

2.4. Proposition. Let $\langle X, \mathcal{A}, \mu \rangle$ be a pure measure space, and let $E \in \mathcal{A}$. Then μ / E is a pure measure.

Proof. Let $\mathcal{B} \subset \mathcal{A}$ be a μ -pure algebra. Since any $E \in \mathcal{A}$ can be written as $E = N_0 \cup \bigcup_{n=1}^{\infty} B_n$ where $\mu N_0 = 0$ and $B_n \in \mathcal{B}_\sigma$ are mutually disjoint (from 1.3 (a)(i)) one may suppose $E \in \mathcal{B}_\sigma$ (in view of 2.3); it will be proved that if this is the case then the algebra $\mathcal{B} / E$ is μ -pure.

Let $D_n \in \mathcal{B} / E$ for $n = 1, 2, \dots$ and $D_n \searrow \emptyset$. There are E_n , $F_n \in \mathcal{B}$ for $n = 1, 2, \dots$ such that $D_n = F_n \cap E$ and $E_n \searrow E$.

Put $A_n = E_n \cap \bigcap_{i=1}^{n} F_i$; then $D_n \subset A_n \in \mathcal{B}$, and $A_n \searrow \emptyset$. Hence $\mu [D_k] \leq \mu [A_k] = 0$ for some k .

2.5. Proposition. Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space. If there exists a μ -pure ring $\mathcal{R}$ then μ is a pure measure.

Proof. One has $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathcal{R}$ are mutually disjoint. Hence the proposition follows from 2.1 (c) and 2.3.

2.6. Lemma ([4]). Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space, let $\mathcal{B}$ be a μ-pure algebra. Then for any countable class $\mathcal{C}_0 \subset \mathcal{A}$ there exists a countable algebra $\mathcal{B}_0 \subset \mathcal{B}$ such that

$$\mu E = \inf \{ \sum_{n=1}^{\infty} \mu B_n \mid B_n \in \mathcal{B}_0 \ \& \bigcup_{n=1}^{\infty} B_n \supset E \}$$

for any $E \in \mathcal{C}_0$.

Proof. For $E \in \mathcal{C}_0$, $n, k = 1, 2, \dots$ there are $B(E, n, k) \in B$ such that

$$E \subset \bigcup_{n=1}^{\infty} B(E, n, k) \quad \text{and} \quad \mu E + \frac{1}{k} > \sum_{n=1}^{\infty} B(E, n, k) .$$

The algebra $\mathcal{B}_0$ spanned by

$$\{ B(E, n, k) \mid E \in \mathcal{C}_0 \ \& \ n, k = 1, 2, \dots \}$$

has the required properties.

2.7. Proposition. Let $\langle X, \mathcal{A}, \mu \rangle$ be a countably-generated measure space. Then the measure μ is pure if and only if it is $\aleph_0$-compact .

Proof. (a) Let μ be pure. It follows from 2.6 (with $\mathcal{C}_0 = \mathcal{A}_0$ from 1.1 (c)) that there exists a countable μ-pure algebra $\mathcal{B}_0$; let $N_1, N_2, \dots$ be all null-sets of $\mathcal{B}_0$. Put $\mathcal{C} = \mathcal{B}_0 / (X \setminus \bigcup_{i=1}^{\infty} N_i)$.

Then $\mu E = \sup \{ \mu C / C \in \mathcal{C}_\delta \ \& \ C \subset E \}$ for any $E \in \mathcal{A}$.

Thus it suffices to show that the class $\mathcal{C}$ is $\aleph_0$-compact (then, obviously, $\mathcal{C}_\delta$ is $\aleph_0$-compact as well). Assume $C_n \in \mathcal{C}$ for $n = 1, 2, \dots$, and $\bigcap_{n=1}^{\infty} C_n = \emptyset$. There are $B_n \in \mathcal{B}_0$ such that $C_n = B_n \setminus \bigcup_{i=1}^{\infty} N_i$. For

$D_n = \bigcap_{i=1}^{n} B_i \setminus \bigcup_{i=1}^{n} N_i$ one has $D_n \searrow \emptyset$ and $D_n \in \mathcal{B}_0$; consequently $\mu D_h = 0$ for some h and $D_h = N_k$ for some k and $D_s = \emptyset$ for $s = max(h, k)$. Hence $\bigcap_{n=1}^{s} C_n = \emptyset$.

(b) Let μ be $\aleph_0$ -compact and $\mathcal{A}_0$ be a countable algebra from 1.1 (c). Then μ is perfect (= quasi-compact, see [6], Th.II); hence there exist mutually disjoint $E_n \in \mathcal{A}$, $n = 1, 2, \dots$, such that $\mathcal{A}_0 / E_n$ are $\aleph_0$ -compact classes and $\mu [\bigcup_{i=1}^{n} E_i] > \mu X - \frac{1}{n}$. Hence all the measures μ / E_n are pure, and $\mu [X \setminus \bigcup_{i=1}^{\infty} E_i] = 0$.

Proposition 2.3 can be applied.

2.8. Corollary. Any pure measure is perfect.

Proof. Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space, let $\mathcal{B}$ be a μ -pure algebra. It is enough to show that the restriction μ_0 of μ to a σ -algebra $\mathcal{A}_0 \subset \mathcal{A}$ is perfect whenever the space $\langle X, \mathcal{A}_0, \mu_0 \rangle$ is countably-generated ([6], Th.III). If this is the case there exists a countable algebra $\mathcal{B}_0 \subset \mathcal{B}$ such that (2.6)

$$\mu E = inf \{ \sum_{n=1}^{\infty} \mu B_n / B_n \in \mathcal{B}_0 \ \& \bigcup_{n=1}^{\infty} B_n \supset E \}$$

for any $E \in \mathcal{A}_0$. Let $\mathcal{A}_1$ be the σ -algebra spanned by $\mathcal{A}_0 \cup \mathcal{B}_0$ and μ_1 be the restriction of μ to $\mathcal{A}_1$; then $\langle X, \mathcal{A}_1, \mu_1 \rangle$ is countably-generated and the measure μ_1 is pure (since the algebra $\mathcal{B}_0$ is μ_1 -pure). By Proposition 2.7 the measure μ_1 is $\aleph_0$ -compact. Hence μ_1 and μ_0 are perfect ([6], Th.III).

2.9. <u>Proposition</u>. Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space, and let the set of values of the measure μ be finite. Then μ is pure.

Proof. One can immediately see that the algebra $\mathcal{A}$ itself is μ-pure.

3. <u>Indirect products</u>

3.1. <u>Notation</u>. Let $\langle X_i, \mathcal{A}_i, \mu_i \rangle$, $i \in I$, be measure spaces such that $\mu_i X_i = 1$. Put $X = \prod_{i \in I} X_i$. Let $\mathcal{A} \subset \exp X$ be the smallest algebra such that $\pi_i^{-1}[\mathcal{A}_i] \subset \mathcal{A}$ for each canonical projection $\pi_i : X \longrightarrow X_i$. Let μ be any positive finitely additive set function on $\mathcal{A}$ such that $\mu(\pi_i^{-1}(E)) = \mu_i E$ for any $i \in I$, and any $E \in A_i$ (μ is often called an indirect product of μ_i's).

3.2. <u>Proposition</u>. Let all the measures μ_i be pure. Then μ is σ-additive and its (unique) σ-additive extension to the σ-algebra spanned by $\mathcal{A}$ is pure.

Proof. There are μ_i-pure algebras $\mathcal{B}_i$; let $\mathcal{B} \subset \subset \exp X$ be the smallest algebra such that $\pi_i^{-1}[\mathcal{B}_i] \subset \mathcal{B}$ for any $i \in I$. We shall show that $\mathcal{B}$ is μ-pure in (a), and conclude the proof in part (b).

(a) Let $B_n \in \mathcal{B}$ for $n = 1, 2, \ldots, B_n \searrow \emptyset$. Assume $\mu B_n > 0$ for all $n = 1, 2, \ldots$. We derive a contradiction as follows. Put $\mathcal{P} = \{\prod_{i \in I} E_i \mid E_i \in B_i$ & there exists a finite set $F \subset I$ such that $E_i = X_i$ for $i \in \in I \setminus F\}$.

Clearly any set in $\mathcal{B}$ is a finite union of sets in $\mathcal{P}$. By induction we shall define sets $P_n \in \mathcal{P}$, $n = 0, 1, 2, \ldots$ such that for any $n = 1, 2, \ldots$ the following three conditions hold:

1) $P_{n-1} \supset P_n$,

2) $P_n \subset B_n$,

3) $\mu(P_n \cap B_k) > 0$ for all $k > n$.

Put $P_0 = X$. If P_i , $i \leq n$ are defined, then $P_n \cap B_{n+1} \in \mathcal{B}$, and hence we may write

$$P_n \cap B_{n+1} = \dot{\bigcup_{s=1}^{r}} R_s$$

with R_s in $\mathcal{P}$. We shall show that $\mu(R_{s_0} \cap B_k) > 0$ for some s_0 and all $k > n$; then we put $P_{n+1} = R_{s_0}$, and Conditions (1) - (3) will be obviously satisfied. If there were no s_0 , then there would exist integers $k(s) > n$, $1 \leq s \leq r$, such that

$$\mu(R_s \cap B_{k(s)}) = 0 ,$$

and hence, for $m = max\{k(s) \mid 1 \leq s \leq r\}$

$$\mu(P_n \cap B_m) = \mu(P_n \cap B_{n+1} \cap \ldots \cap B_m) = 0 ,$$

which would contradict Condition (3).

Now let $\{P_n\}$ be any sequence in $\mathcal{P}$ which satisfies Conditions (1) - (3) above.

It follows that $\pi_i[P_n] \searrow^n \emptyset$ for some $i \in I$ and $\mu_i(\pi_i[P_h]) = 0$ for some h because $\pi_i[P_h] \in B_i$. But $\mu(P_h) \leq \mu(\pi_i^{-1}(\pi_i[P_h])) = \mu_i(\pi_i[P_h]) = 0$ which is the required contradiction.

(b) For any $E \subset X$ put

$$\mu^* E = \inf \{ \sum_{m=1}^{\infty} \mu B_m \,/\, B_m \in \mathcal{B} \ \& \bigcup_{m=1}^{\infty} B_m \supset E \} \quad .$$

In the part (a) of the proof it was shown that μ is σ-additive on $\mathcal{B}$ (and so $\mu E = \mu^* E$ for any $E \in \mathcal{B}$).

In order to finish the proof it is only to show that $\mu E = \mu^* E$ for any $E \in \mathcal{A}$ ([1], 12.c).

Firstly, let $E = \prod_{i \in F} E_i \times \prod_{i \in I \setminus F} X_i$, where $F \subset I$ is finite and $E_i \in \mathcal{A}_i$ for $i \in F$. Let ε be any positive real number. For any $i \in F$ there exist mutually disjoint $B(i, m) \in \mathcal{B}_i$ such that

$$E_i \subset \bigcup_{m=1}^{\infty} B(i, m)$$

and $\sum_{m=1}^{\infty} \mu_i (B(i, m)) < \mu_i E_i + \varepsilon$

(i.e. $\sum_{m=1}^{\infty} \mu_i (B(i, m) \setminus E_i) < \varepsilon$).

Further, $\bigcup_{x \in N^F} (\prod_{i \in F} B(i, x(i)) \times \prod_{i \in I \setminus F} X_i) \supset E$ and

$$\sum_{x \in N^F} \mu (\prod_{i \in F} B(i, x(i)) \times \prod_{i \in I \setminus F} X_i) =$$

$$= \sum_{x \in N^F} \mu [(\prod_{i \in F} B(i, x(i)) \times \prod_{i \in I \setminus F} X_i) \cap E) +$$

$$+ \sum_{x \in N^F} \mu [(\prod_{i \in F} B(i, x(i)) \times \prod_{i \in I \setminus F} X_i) \setminus E) \leq$$

$$\leq \mu E + \sum_{x \in N^F} \mu [\bigcup_{j \in F} (B(j, x(j)) \setminus E_j) \times$$

$$\times \prod_{i \in F \setminus \{j\}} B(i, x(i)) \times \prod_{i \in I \setminus F} X_i] \leq$$

$$\leq \mu E + \sum_{x \in N^F} \sum_{j \in F} \mu [(B(j, x(j)) \setminus E_j) \times$$

$$\times \prod_{i\in F\setminus\{j\}} B(i,x(i)) \times \prod_{i\in I\setminus F} X_i] =$$

$$= \mu E + \sum_{j\in F} \sum_{n=1}^{\infty} \sum_{x\in N^{F\setminus\{j\}}} \mu [(B(j,n)\setminus E_j) \times$$

$$\times \prod_{i\in F\setminus\{j\}} B(i,x(i)) \times \prod_{i\in I\setminus F} X_i] \leq \mu E +$$

$$+ \sum_{j\in F} \sum_{n=1}^{\infty} \mu [(B(j,n)\setminus E_j) \times \prod_{i\in I\setminus\{j\}} X_i] =$$

$$= \mu E + \sum_{j\in F} \sum_{n=1}^{\infty} \mu_j (B(j,n)\setminus E_j) < \mu E + \varepsilon \cdot \mathrm{card}\, F$$

because μ is finitely additive,

$$(\prod_{i\in F} B(i,x(i)) \times \prod_{i\in I\setminus F} X_i) \setminus E =$$

$$= \bigcup_{j\in F} (B(j,x(j))\setminus E_j) \times \prod_{i\in F\setminus\{j\}} B(i,x(i)) \times \prod_{i\in I\setminus F} X_i$$

and

$$\bigcup_{x\in N^{F\setminus\{j\}}} (B(j,n)\setminus E_j) \times \prod_{i\in F\setminus\{j\}} B(i,x(i)) \times \prod_{i\in I\setminus F} X_i \subset$$

$$\subset (B(j,n)\setminus E_j) \times \prod_{i\in I\setminus\{j\}} X_i \quad .$$

This shows that $\mu^* E \leq \mu E$ for any $E = \prod_{i\in F} E_i \times \times \prod_{i\in I\setminus F} X_i$. But any set in $\mathcal{A}$ is a disjoint finite union of such sets; hence $\mu^* E \leq \mu E$ for all $E \in \mathcal{A}$ and

$$\mu E = \mu X - \mu (X\setminus E) \leq \mu^* X - \mu^* (X\setminus E) \leq \mu^* E$$

for all $E \in \mathcal{A}$.

4. Purely $\aleph_0$ -compact measures

4.1. Definition. (a) If $\langle X, \mathcal{A}, \mu \rangle$ is a measure space then a ring $\mathcal{R} \subset \mathcal{A}$ is μ -purely $\aleph_0$ -compact if (i) $\mu E = \inf \{ \sum_{m=1}^{\infty} \mu B_m / B_m \in \mathcal{R} \ \& \ \bigcup B_m \supset E \}$ for any $E \in \mathcal{A}$ and (ii) $B_m \in \mathcal{R}$ for $m = 1, 2, \dots$, $B_m \searrow \emptyset$ imply $B_h = \emptyset$ for some h.

(b) Measure μ is purely $\aleph_0$ -compact if there exists a μ -purely $\aleph_0$ -compact algebra.

4.2. Remarks. (a) The condition (ii) in 4.1 shows that the ring $\mathcal{R}$ is $\aleph_0$ -compact (in the sense of 1.2), hence $\mathcal{R}_\delta$ is $\aleph_0$ -compact as well; moreover, Condition (i) gives $\mu E = \sup \{ \mu R / R \in \mathcal{R}_\delta \ \& \ R \subset E \}$ for any $E \in \mathcal{A}$.
Consequently, every purely $\aleph_0$ -compact measure is $\aleph_0$ -cpmpact.

(b) Obviously, every purely $\aleph_0$ -compact measure is pure. We do not know whether every pure measure is purely $\aleph_0$ -compact and whether every $\aleph_0$ -compact measure is purely $\aleph_0$ -compact, even for two-valued measures.

(c) All propositions but one of Sections 2,3 hold if pure is replaced by purely $\aleph_0$ -compact.

The proofs work without any essential change. The only exception is Proposition 2.9. We conjecture that it does not hold for purely $\aleph_0$ -compact measures, i.e. that there is a two-valued measure that is not purely $\aleph_0$ -compact.

This problem is closely related to those in (b) since any two-valued measure is $\aleph_0$ -compact.

(d) Assume that $\langle X, \mathcal{A}, \mu \rangle$ is a complete measure space, and let $Y \subset X$ support μ, i.e. $\mu A = \bar{\mu}(A \cap Y)$ for each A in $\mathcal{A}$. Let $\mathcal{A}'$ be the collection of $A \cap Y$, $A \in \mathcal{A}$, and let μ' be the measure on $\mathcal{A}'$ defined by $\mu'(A \cap Y) = \mu A$. If $\langle Y, \mathcal{A}', \mu' \rangle$ is pure then so is $\langle X, \mathcal{A}, \mu \rangle$. Indeed, if $\mathcal{B}'$ is μ'-pure algebra, let $\mathcal{B}$ be the collection of all $B \in \mathcal{A}$ such that $B \cap Y \in \mathcal{B}'$. The idea of this remark will be developed elsewhere.

§ 5. Stone spaces

In this short paragraph $\aleph_0$-compact and μ-pure algebras will be characterized by means of the topological and measure properties of the remainder in the Stone space.

Let $\langle X, \mathcal{B} \rangle$ be a measurable space, i.e. a set X endowed with an algebra $\mathcal{B}$ of subsets of X. For simplicity we shall assume that the elements of $\mathcal{B}$ separate the points of X. Then X may be regarded to be a subset of the Stone space $\mathcal{X}$ of the Boolean algebra $\mathcal{B}$. Recall that $\mathcal{X}$ is uniquely determined (up to an isomorphism having fixed the points of X) by the following properties:

a. $\mathcal{X}$ is a compact Hausdorff space such that the clopen sets (the sets which are simultaneously closed and open) form a basis for open sets.

b. X is a dense subset of $\mathcal{X}$.

c. A set $B \subset X$ belongs to $\mathcal{B}$ if and only if $B = X \cap G$ for some clopen set G in $\mathcal{X}$ (and then G is the closure of B in $\mathcal{X}$).

A subset Y of a topological space Z is <u>G_δ-dense</u> if

$Z \setminus Y$ contains no non-void G_δ -set. Evidently, if Y is G_δ -dense in Z, then Y is dense in Z (and the converse need not be true).

5.1. Theorem. $\mathcal{B}$ is $\aleph_0$ -compact if and only if X is G_δ -dense in the Stone space K of $\mathcal{B}$.

Proof. If $\mathcal{B}$ is not $\aleph_0$ -compact, then $B_n \searrow \emptyset$ for some non-void B_n in $\mathcal{B}$, and then

$$G = \bigcap \{\overline{B}_n\}$$

is a non-void G_δ set in $K \setminus X$.

Conversely, assume that G is a non-void G_δ in $K \setminus X$. Pick any x in G, and choose B_n in $\mathcal{B}$ such that $x \in \overline{B}_n \subset G_n$, where G_n are open in K and G is the intersection of $\{G_n\}$. Then $\bigcap \{B_n\} = \emptyset$, however

$$C_k = \bigcap \{B_n \,/\, n \leq k\}$$

is non-void for each k because

$$\overline{C}_k = \bigcap \{\overline{B}_n \,/\, n \leq k\}$$

is a neighborhood of x in K, and X is dense in K.

It may be of certain interest to look on $\aleph_0$ -compact algebras from the point of view of uniform spaces. Every algebra $\mathcal{B}$ on X defines a precompact uniformity $u_{\aleph_0}\mathcal{B}$ which has finite partitions of X by elements of $\mathcal{B}$ for a basis for uniform covers. In fact, the Stone space of $\mathcal{B}$ is a completion of $u_{\aleph_0}\mathcal{B}$. The following result is easy to prove.

5.2. Theorem. The following properties of a uniform space are equivalent:

1. Z precompact and G_δ -dense in its completion.

2. If Z_n are zero sets in Z, $\bigcap Z_n = \emptyset$, then $\bigcap \{Z_n / n \leq k\} = \emptyset$ for some k.

3. Every uniformly continuous function on Z assumes its infimum (and supremum).

4. Z is a precompact inversion-closed uniform space.

Perhaps the uniform spaces with the properties in the preceding theorem should be called pseudocompact. Thus $\mathcal{B}$ is $\aleph_0$-compact if and only if $u_{\aleph_0} \mathcal{B}$ is pseudocompact.

5.3. Remark. For uniform methods in measurable spaces see "Topological methods in measure and measurable spaces", Proc. Third Prague Topological Symposium, Academia (Prague 1972) or Academic Press (1972). For a development of the theory of uniform spaces relevant to measure and measurable spaces we refer to Z. Frolík, A. Hager: "Maps of uniform spaces", in preparation.

If μ is a measure on $\mathcal{B}$, then one can define a measure $\hat{\mu}$ on clopen sets in K by setting

$$\hat{\mu} B = \mu (B \cap X) .$$

Then $\hat{\mu}$ extends to a regular Borel measure on K, and μ is σ-additive if and only if the inner $\hat{\mu}$-measure of $K \setminus X$ is zero (that means, if $C \subset K \setminus X$ is compact then $\hat{\mu} C = 0$).

5.4. Theorem. $\mathcal{B}$ is μ-pure if and only if the following condition is satisfied:

if $C \subset K \setminus X$ is compact G_δ then there exists an open set $G \supset C$ such that $\hat{\mu} G = 0$ (or equivalently, there exists a clopen set $G_0 \supset C$ such that $\hat{\mu} G_0 = 0$).

Proof. Assume that $B_n \searrow \emptyset$, $B_n \in \mathcal{B}$ and $\mu B_n > 0$ for each n. Put $C = \bigcap \{\overline{B}_n\}$; C is compact G_δ. Each open $G \supset C$ contains some $\overline{B}_n$ and the condition is not satisfied.

Conversely, assume that $\mathcal{B}$ is μ-pure and let $C \subset \subset \hat{X} \setminus X$ be a compact G_δ set. Choose a decreasing sequence $\{C_n\}$ of clopen sets such that $C_n \searrow C$. Then $C_n \cap \cap X \searrow \emptyset$. Hence $\hat{\mu}\, C_k = \mu(C_k \cap X) = 0$ for some k. This concludes the proof.

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Matematický ústav ČSAV
Praha 1, Žitná 25
Československo

Matematicko-fyzikální
fakulta Karlovy university
Sokolovská 83, Praha 8
Československo

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