Jules C. Varlet
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NODAL FILTERS IN SEMILATTICES

J.C. VARLET, Liège

Abstract: A filter of a semilattice S is said to be nodal if it is comparable with any filter of S in the set of all filters of S ordered by inclusion. The nodal filters of S form a chain and induce a partition of S to which an interesting congruence is associated. Moreover, the Dedekind cut of a nodal filter is again a nodal filter.

Nodal filters have especially nice properties in implicative semilattices, i.e. semilattices on which a second binary operation $*$ is defined. We characterize nodal filters solely by means of the latter operation. We also determine the sublagebras which are in direct connection with nodal filters and, by the way, we focus our attention on the irreducible elements of the semilattice. Finally we obtain a characterization of the nodal filters in terms of congruences.

Key words: congruence, endomorphism, filter, implicative semilattice, irreducible, lower semilattice, node.

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§ 0. Preliminaries

The word semilattice will always mean lower semillatice, i.e. a commutative idempotent semigroup or, equivalently, a partially ordered set (abbreviated poset) in which any two elements a and b have a greatest lower bound, denoted by $a \cdot b$ or simply ab, the partial ordering being defined by $a \leq b$ if and only if $ab = a$. The least and greatest elements of a semilattice S, when they exist, will be denoted by 0 and 1 respectively. When S is a lattice, the second binary operation will be denoted by $+$.

The symbols $\cap$, $\cup$, $-$, $\subseteq$ and $\subset$ will be used in their usual set-theoretical meaning: intersection, union, difference, inclusion and strict inclusion.

A filter of a semilattice S is a non-empty subset F of S such that $xy \in F$ if and only if $x \in F$ and $y \in F$. The principal filter generated by an element a of S, i.e. the set $\{x : x \in S, x \geq a\}$, will be denoted by $[a)$.

When ordered by inclusion, the set $\mathcal{F}(S)$ of all filters of an up-directed semilattice S is a lattice in which, for any $F, G \in \mathcal{F}(S)$, $F \cdot G = F \cap G$ and $F + G$ is the filter generated by $F \cup G$. Of course, if S is not directed above, $F \cap G$ is a filter only if non-empty.

An element a of a semilattice is irreducible if $a = bc$ implies $a = b$ or $a = c$.

A semilattice S is implicative if, for any $a, b \in S$, there exists in S a (unique) element $a * b$ such that $ax \leq b$ if and only if $x \leq a * b$. Hence any implicative semilattice can be considered as an algebra $\mathcal{S} = \langle S; \cdot , * \rangle$ of type $\langle 2, 2 \rangle$. An implicative semilattice is distributive and always has a greatest element 1 .

Terminology and notations are mainly borrowed from [4].

§ 1. Nodal filters in arbitrary semilattices

In [1] R. Balbes and A. Horn have introduced the notion of node in the context of a lattice but it makes sense in any poset. A node of a poset S is an element which is comparable with every element of S . We are going to generalize this concept but we first need a lemma.

Lemma 1.1. For a filter F of a semilattice S, the following conditions are equivalent:

(1) for every $x \in F$ and every $y \notin F$, the relation $x > y$ is satisfied;

(2) for any filter G of S, either $G \subseteq F$ or $G \supseteq F$;

(3) F is a node of $\mathcal{F}(S)$.

Proof. (1) $\Longrightarrow$ (2). Let us suppose there exists a filter G incomparable with F. Then there are elements x and y such that $x \in F - G$, $y \in G - F$ and $x \not> y$.

(2) $\Longrightarrow$ (3). Immediate by the definition of a node.

(3) $\Longrightarrow$ (1). If F is a node of $\mathcal{F}(S)$, then for every $x \in F$ and every $y \notin F$ we have $[y) \nsubseteq F$, hence $[y) \supset \supset F \supset [x)$ and $x > y$.

Definition 1.2. A filter satisfying one of the conditions (1) - (3) will be called a nodal filter.

Trivially the whole semilattice S is an improper nodal filter. A principal filter $[x)$ is nodal if and only if x is a node. Two nodal filters are always comparable. These observations are summarized in the following statement.

Lemma 1.3. The set $\mathcal{N}(S)$ of all nodal filters of a semilattice S, ordered by inclusion, is a chain whose greatest element is S. It has not necessarily a least element; nevertheless, if S has an element 1, then $\mathcal{N}(S)$ has the least element $[1)$.

Note that $\mathcal{N}(S)$ can have a least element even when S is not bounded above. In a chain all filters are nodal; on the contrary, some semilattices have no proper nodal filters (take for instance the direct product of two chains isomorphic to the set of integers).

In a lattice, any proper non-principal nodal filter is prime.

In a semilattice S with 0, any proper nodal filter is contained in $\mathcal{D}(S)$, the dense set of S. In fact, let us suppose that the proper nodal filter F of S contains a non-dense element a. Hence there is $b \neq 0$ such that $ab = 0$ and $b \notin F$, an impossibility since $a > b$.

If, in a semilattice S with 0, $\mathcal{D}(S)$ is a principal filter, we can form $\mathcal{D}(\mathcal{D}(S)) = \mathcal{D}^2(S)$. Let us now consider a semilattice S with 0 in which $\mathcal{D}(S), \mathcal{D}^2(S), \dots$ $\dots, \mathcal{D}^n(S)$ form a finite sequence of principal filters. We can claim that all proper principal nodal filters of S belong to this sequence; their generating elements are exactly the nodes of S.

<u>Definition 1.4</u>. We say that two elements x and y of a semilattice S are <u>connected</u> (in symbols, $(x, y) \in \mathcal{R}$) if there is no nodal filter which separates them. Let us notice that $(x, y) \notin \mathcal{R}$ implies either $x > y$ or $x < y$.

<u>Theorem 1.5</u>. In any semilattice S, the relation $\mathcal{R}$ enjoys the following properties:

(1) $\mathcal{R}$ is a congruence of $\mathcal{S} = \langle S ; \cdot \rangle$;

(2) any $\mathcal{R}$-class contains at most one node;

(3) an $\mathcal{R}$-class is totally ordered if and only if it is a singleton;

(4) $S/\mathcal{R}$ is a chain dually isomorphic to $\mathcal{N}(S)$.

Proof. (1) $(x, y) \in R$, $(y, z) \in R$ and $(x, z) \notin R$ are incompatible since, by the latter, there is a nodal filter F such that, for instance, $x \in F$ and $z \notin F$. We then have $y \in F$ and $(y, z) \notin R$, a contradiction. Thus R is an equivalence relation on S. Moreover, if $(x, y) \in R$ then $(xs, ys) \in R$ for every $s \in S$ since otherwise $xs \in F$ and $ys \notin F$ for some nodal filter F, hence $x \in F$, $s \in F$ and $y \notin F$, which contradicts $(x, y) \in$ $\in R$.

(2) Let a and b be connected nodes of S. We have either $a < b$ or $b < a$. In the first case, for instance, a and b are separated by the nodal filter $[b)$.

(3) Let $[a]R$, the R-class of a, be totally ordered and $(b, a) \in R$, $b \neq a$. Any $x \in S$ is comparable with a and b. Hence both a and b are nodes, which contradicts (2).

(4) Let us define the mapping $\alpha : S/R \longrightarrow \mathcal{N}(S)$ by $C\alpha =$ $= F_C$, where F_C is the nodal filter generated by the R-class C. In fact, $F_C = \{x \in S : x \geq y,\ y \in C\}$. Obviously α is bijective and $C \leq C'$ in S/R if and only if $F_C \geq F_{C'}$ in $\mathcal{N}(S)$.

In [8] we defined, for any element a of the semilattice S, the subset D_a as follows: $D_a = \{x \in S : xy \leq a$ implies $y \leq a\}$. It is clear that D_a is a filter if non-empty. The following theorem provides us with a new characterization of nodal filters.

Theorem 1.6. A non-empty subset F of a semilattice S is a nodal filter if and only if $F = \bigcap\{D_x : x \notin F\}$.

Proof. 1°) if: since the set-intersection of filters is a filter if non-empty, we just have to prove that F is nodal. If not, there exist $y \in F$ and $z \notin F$ with y and z incomparable, hence $yz = x < z$. Since $y \in D_x$, $yz = x$ implies $z \leq x$, a contradiction.

2°) only if: let F be a nodal filter of S. For every $y \in F$ and every $x \notin F$, we have $y > x$ and $yz \leqq x$ implies $z \leq x$, hence $y \in D_x$ and $F \subseteq D_x$. Since $x \notin D_x$ for any $x \neq 1$, the proof is complete.

Now we direct our attention to the Mac Neille completion of the semilattice S or, more precisely, to the dual of the latter. It means that to every subset A of S we associate its "Dedekind cut" $(A^{l})^{u}$, i.e. all uper bounds to the set of lower bounds of A.

We call a filter F of S <u>normal</u> if $F = (F^{l})^{u}$. Obviously any principal filter is normal. The normality of a non-principal nodal filter can be characterized as follows.

<u>Theorem 1.7.</u> In a semilattice S, for a non-principal nodal filter F, the following aonditions are equivalent:

(1) F is normal;

(2) $S - F$ is not a principal ideal;

(3) $\inf F$ does not exist.

<u>Proof.</u> (1) $\Longrightarrow$ (2). First, let us observe that for any non-principal nodal filter F holds $S - F = F^{l}$. If $S - F = (a]$, then $(F^{l})^{u} = [a) \supset F$ and F is not normal.

(2) $\Longrightarrow$ (3). If $\inf F = a$, then $a \notin F$, a is the greatest element of $S - F$, hence $S - F$ is a principal ideal.

(3) $\Longrightarrow$ (1). If F is not normal, then $(F^1)^u \supset F$ and there is an element $a \notin F$ which is an upper bound of $S - F$. This element obviously constitutes the greatest element of $S - F$. It is also the infimum of F.

Corollary 1.8. In a semilattice S, if a filter F is nodal, then its Dedekind cut $(F^1)^u$ is also nodal.

Proof. Since the case F principal is trivial, by virtue of the preceding theorem we may restrict ourselves to the consideration of a non-principal nodal filter F for which $\inf F$ exists. Let $a = \inf F$, thus $a \notin F$. For any $x \in S - F$ we have $x \leq a$, a is a node whence $[a) = (F^1)^u$ is nodal.

§ 2. Nodal filters in implicative semilattices

First of all we characterize the nodes and the nodal filters of an implicative semilattice by means of the only binary operation $*$.

Theorem 2.1. Each of the following two conditions is necessary and sufficient for an element a of an implicative semilattice S to be a node:

(1) for every $x \in S$, either $a * x = 1$ or $x * a = 1$;

(2) for every $x \in S$, $a * x = x$ or 1.

Proof. It is obvious for (1) since $x * y = 1$ if and only if $x \leq y$.

Now if a is a node, then $x \geq a$ implies $a * x = 1$ and $x < a$ gives $a * x = x$. If a is not a node, there exists b not comparable with a. Then $ab < a$ and

$a * ab \neq 1$; moreover $a * ab \geq b$, hence $a * ab \neq ab$.

We remind the reader ([5], p.63) that a subset F of an implicative semilattice S is a filter if and only if

(i) $1 \in F$;

(ii) $a \in F$ and $a * b \in F$ imply $b \in F$.

The proof is given for implicative lattices but no use is made of the second lattice-operation.

<u>Theorem 2.2</u>. A subset F of an implicative semilattice S is a nodal filter if and only if

(i′) $1 \in F$;

(ii′) $a * b = 1$ and $a \in F$ imply $b \in F$;

(iii′) $a * b \neq 1$ and $a * b \in F$ imply $a, b \in F$.

<u>Proof</u>. 1°) if: since the system (i′) - (iii′) is obviously stronger than (i) - (ii), F is a filter. It remains to prove that F is nodal. If not, there exist $a \notin F$ and $b \in F$ such that $a \nleq b$. Then $a * b \neq 1$ and $a * b \in F$ (owing to $a * b \geq b$). By (iii′) $a \in F$, which is a contradiction.

2°) only if: since $F \neq \emptyset$, $F \ni 1$. Since $a * b = 1$ is equivalent to $a \leq b$, (ii′) holds in any filter. To prove (iii′), let us assume $a * b \neq 1$ together with $a * b \in F$ and consider three cases.

Case 1. $a \notin F$ and $b \notin F$. Then $a * b \begin{cases} = 1 & \text{if } a \leq b \\ \notin F & \text{otherwise.} \end{cases}$

Case 2. $a \in F$ and $b \notin F$. Then $a * b = b$.

Case 3. $a \notin F$ and $b \in F$. Then $a * b = 1$.

In all these cases one of the premises is violated, hence the only possibility is $a \in F$ and $b \in F$.

Our next concern will be the determination of some subalgebras of the implicative algebra $\mathcal{S} = \langle S ; \cdot , * \rangle$. Clearly any filter of S is a subalgebra. Less obvious is the following proposition, in which R has the same meaning as in 1.4.

<u>Theorem 2.3</u>. In an implicative algebra $\mathcal{S} = \langle S ; \cdot , * \rangle$, for any subset A of S, $B = (\bigcup\{[x]R ; x \in A\}) \cup \{1\}$ is a subalgebra.

<u>Proof</u>. For any $y, z \in B$, $yz \in B$ and $y * z = 1$ if $y \leq z$, whereas $y * z \in [z]R$ if $y \nleq z$. Only the last assertion is worth explaining. If $(y, z) \notin R$ and $y > z$, then $y * z = z$. Let us suppose now $(y, z) \in$ $\in R$ and $y \nleq z$. Since $y * z \geq z$ always holds, if $(y * z, z) \notin R$, then there exists a nodal filter F containing $y * z$ but not z . As $(y, z) \in R$, $F \not\ni y$ hence $y * z > y$ and $y(y * z) = y$. But, by definition of $y * z$, we have $y(y * z) \leq z$. This leads to the contradiction $y \leq z$, q.e.d.

If $b > a$ and $(a, b) \notin R$, then clearly $b * a = a$. But we can have $b * a = a$ even when $b \ngtr a$ and $(a, b) \in$ $\in R$. Before giving a necessary and sufficient condition ensuring the previous equality, we introduce a definition.

<u>Definition 2.4</u>. An element a of the semilattice S will be said <u>irreducible with respect to</u> b ($b > a$) if $bx = a$ implies $x = a$. Then a is also irreducible with respect to any $c \in S$ such that $c \geq b$. Let us notice that an

element a is irreducible if and only if it is irreducible with respect to any $b > a$.

Theorem 2.5. In an implicative semilattice S, $b * a = a$ if and only if a is irreducible with respect to any upper bound of $\{a, b\}$.

Proof. 1°) if: we have to show that $b * a = a$, i.e. $bx \leq a$ if and only if $x \leq a$. Only the direct implication is not trivial. By virtue of the distributivity of S , $bx \leq a$ implies the existence of elements b_1 and x_1 satisfying $b_1 \geq b$, $x_1 \geq x$ and $b_1 x_1 = a$. The element b_1 is an upper bound of $\{a, b\}$, hence $x_1 = a$ and $x \leq a$.

2°) only if: let x be an upper bound of $\{a, b\}$. We have to show that $xy = a$ implies $y = a$. Clearly it suffices to prove that $y \leq a$. Since $x \geq b$, one has $x * a \leq b * a = a$, hence $x * a = a$ ($x * a \geq a$ always holds), and $xy = a$ implies $y \leq a$.

Corollary 2.6. Let a and b be two elements of the implicative semilattice S such that $b > a$. Then $b * a = a$ if and only if a is irreducible with respect to b .

Corollary 2.7. In an implicative lattice L , $b * a = a$ if and only if a is irreducible with respect to $a + b$. ([7], Theorem 4)

Theorem 2.8. In an implicative semilattice S , a chain C is a subalgebra if and only if

(1) $C \ni 1$;

(2) any $x \in C$ is irreducible with respect to its succes-

sors in C .

Proof. 1^o) if: let x, y $(x < y)$ be any two elements of the chain C . Since $x * x = y * y = x * y = 1 \in C$ and $y * x = x$ by Corollary 2.6, C is a subalgebra.
2^o) only if: let C be a totally ordered subalgebra of S . Clearly C has to contain 1 . If $x \in C$, $y \in C$ and $x < y$, then $y * x = x$ since $y * x$ has to belong to C and $y * x \geq x$.

Remark 2.9. In an implicative semilattice S , a chain C is a subalgebra if, for any $x, y \in C$ $(x < y)$, either $(x,y) \notin R$ or x is irreducible. For instance, the set of all nodes of S is a subalgebra. So it is interesting to characterize irreducible elements of an implicative semilattice. Such a work was done in [6] and [7], but in the context of lattices.

Theorem 2.10. In an implicative semilattice S , an element a is irreducible if and only if $x \nleq a$ implies $x * a = a$.

Proof. 1^o) if: we have to show that $yz = a$ implies $y = a$ or $z = a$. From $yz = a$ follows $z \leq y * a$. Let us suppose $y \neq a$. Since $y < a$ is impossible, the condition $y \nleq a$ is satisfied and the hypothesis yields $y * a = a$, hence $z = a$.
2^o) only if: let a be irreducible. For any $x \nleq a$, a is irreducible with respect to any upper bound of $\{x, a\}$, hence, by Theorem 2.5, $x * a = a$.

To end with we shall characterize nodal filters in terms of congruences. But here also we need some preliminaries.

Hereafter $End(S)$ will mean the endomorphism monoid of the implicative algebra $\mathcal{S} = \langle S; \cdot, * \rangle$. To every endomorphism α of $\mathcal{S}$ is associated a congruence Θ_α defined by

$$(x,y) \in \Theta_\alpha \quad \text{if and only if } x\alpha = y\alpha .$$

Let us recall that an endomorphism α of an algebra $\mathcal{A} = \langle A; F \rangle$ is said to be a <u>left vector endomorphism</u> [3] if there exists a subalgebra $\mathcal{B} = \langle B; F \rangle$ of $\mathcal{A}$ satisfying the following two conditions:

(1) $\bigcup \{ [x]\Theta_\alpha : [x]\Theta_\alpha \cap B \neq \emptyset , x \in A \} = A$, i.e. the uniton of the Θ_α-classes which meet B is A;

(2) $\Theta_\alpha | B = \omega_B$, where ω_B is the equality on B (in Cohn's terminology ([2], p.59), B is a transversal for $\mathcal{A}/\Theta_\alpha$ in $\mathcal{A}$).

We finally remind the reader that, for any congruence Θ of the implicative semilattice $\mathcal{S}$, [1] Θ is a filter of S; we shall denote it by F_Θ. Moreover $(x,y) \in \Theta$ if and only if $xd = yd$ for a suitable $d \in F_\Theta$. Conversely, if F is any filter of S, then the relation Θ_F defined by

$$(x,y) \in \Theta_F \quad \text{if and only if } xd = yd \text{ for}$$
$$\text{some } d \in F$$

is a congruence. In other words, the correspondence between filters and congruences is one-to-one.

When $[1]\Theta$ is a nodal filter, the corresponding congruence is rather special, as shown by the following theorem.

Theorem 2.11. In an implicative semilattice $\mathcal{S} = \langle S; \cdot, * \rangle$ for any congruence Θ, the following three conditions are equivalent:

(1) $[1]\Theta$ is a nodal filter;

(2) Θ is a node of $Con(\mathcal{S})$, the congruence lattice of $\mathcal{S}$;

(3) for every $x \notin [1]\Theta$, $[x]\Theta = \{x\}$.

Moreover, for any congruence Θ of $\mathcal{S}$ satisfying these conditions, there exists $\alpha \in End(\mathcal{S})$ such that $\Theta_\alpha = \Theta$.

Proof. First let us observe that the equivalence of (1) and (2) is obvious: the mapping $\Theta \longrightarrow F_\Theta$ of $Con(\mathcal{S})$ onto $\mathcal{F}(S)$ is an isomorphism and F_Θ is a nodal filter if and only if it is a node of $\mathcal{F}(S)$.

(1) implies (3) since, for every $x \notin [1]\Theta$ and every $y \in [1]\Theta$, we have $xy = x$, hence $[x]\Theta = \{x\}$.

(3) implies (1). Let us suppose there exist $f \in F_\Theta$ and $a \notin F_\Theta$ such that $f \not> a$. Then $fa \neq a$ and, however, since $(fa)f = af$, $(fa, a) \in \Theta$: $[a]\Theta$ would no longer be a singleton.

Finally, if Θ is a congruence such that $[1]\Theta$ is a nodal filter F, the mapping α of S into S defined by

$$x\alpha = \begin{cases} 1 & \text{if } x \in F; \\ x & \text{otherwise} \end{cases}$$

is an endomorphism and $\Theta_\alpha = \Theta$. In fact, it is routine to check that α preserves the two binary operations in all possible cases as in the proof of Theorem 2.2.

Corollary 2.12. In an implicative algebra $\mathcal{S} = \langle S; \cdot, * \rangle$, all endomorphisms α for which $[1]\Theta_\alpha$ is a nodal filter, are left vector endomorphisms.

Proof. Thanks to Theorems 2.3 and 2.11, we can claim that $(S - [1]\Theta_\alpha) \cup \{1\}$ is the required subalgebra.

References

[1] R. BALBES and A. HORN: Injective and projective Heyting algebras, Trans.Amer.Math.Soc.148(1970),549-559.

[2] P. COHN: Universal Algebra, Harper and Row, 1965.

[3] P. DUBREIL: Endomorphismes. Séminaire Dubreil-Pisot,Paris,1964/65, No 23.

[4] G. GRÄTZER: Universal Algebra, Van Nostrand, 1968.

[5] H. RASIOWA and R. SIKORSKI: The mathematics of metamathematics, Monogr.Mat.,t.41,PWN,Warsaw, 1963.

[6] D. SMITH: Meet-irreducible elements in implicative lattices, Proc.Amer.Math.Soc.34(1972),57-62.

[7] J. VARLET: Contribution à l'étude des treillis pseudo-complémentés et des treillis de Stone, Mémoires Soc.Roy.Sc.Liège,t.8,fasc.4(1962),1-71.

[8] J. VARLET: Relative annihilators in semilattices, to appear in Bull.Austral.Math.Soc.

Institut de Mathématique
Université de Liège
B-4000 Liège
Belgique

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