Commentationes Mathematicae Universitatis Carolinae

Donald Joseph Hansen

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Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 2, 231--239

Persistent URL: http://dml.cz/dmlcz/105488

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14,2 (1973)

COMPATIBLE PARTIAL ORDERINGS IN BOOLEAN ALGEBRAS

D.J. HANSEN, Raleigh

Abstract: A compatible partial ordering $\leq$ in a Boolean algebra $(B, +, \cdot)$ is a partial ordering defined on the set B such that $a + x \leq b + x$ and $a \cdot x \leq b \cdot x$ for each a, b and x in B with $a \leq b$. It is proved that if a compatible poset $(B, \leq)$ contains a pair of comparable minimal and maximal elements then $(B, \leq)$ is isomorphic to the cardinal sum of a family of isomorphic Boolean lattices. Also, it will be shown that the condition of finiteness on a Boolean algebra $(B, +, \cdot)$ is necessary and sufficient in order for each of its compatible posets $(B, \leq)$ to have a structure of the above form. Last, it is proved that the number of compatible posets which can be constructed in a finite Boolean algebra of cardinality 2^n is 3^n.

Key-words and phrases: poset, cardinal sum of posets, Boolean lattice, Boolean algebra, compatible partial ordering in a Boolean algebra.

AMS, Primary: 06A40 Ref. Ž. 2.724.2
Secondary: 06A75

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1. Introduction. A compatible partial ordering $\leq$ in a Boolean algebra $(B, +, \cdot)$ is a partial ordering defined on the set B such that $a + x \leq b + x$ and $a \cdot x \leq b \cdot x$ for each a, b and x in B with $a \leq b$. Well known examples of such orderings are $R_1(\leq) = \{(x, y) \mid x \in B, y \in B,$

and $x \cdot y = x \}$, the dual of $R_1(\leq)$, and the trivial ordering $R_2(\leq) = \{(x,x) | x \in B\}$.

The objective of this paper is to study the structure of those compatible posets $(B, \leq)$ which possess at least one pair of comparable minimal and maximal elements. It will be shown that such a poset is isomorphic to the cardinal sum of a family of isomorphic Boolean lattices. Also, it will be proved that the condition of finiteness on a Boolean algebra $(B, +, \cdot)$ is necessary and sufficient in order for each of its compatible posets $(B, \leq)$ to have a structure of the above form. Last, it will be shown that the number of compatible posets which can be constructed in a finite Boolean algebra of cardinality 2^n is 3^n.

Throughout $(B, +, \cdot)$ will denote a Boolean algebra with 0 and 1 denoting respectively the additive and multiplicative identity in B. Also, for each x in B, x^* will denote the complement of x in B.

Terminology and background material needed for this article may be found in [1].

2. <u>Main results</u>.

<u>Lemma 1</u>. If $\leq$ is a compatible partial ordering in $(B, +, \cdot)$ and each of m_1 and m_2 is a maximal (minimal) element in B, with $1 \leq m_1$ and $1 \leq m_2$ ($m_1 \leq 1$ and $m_2 \leq 1$) then $m_1 = m_2$.

<u>Lemma 2</u>. Suppose $\leq$ is a compatible partial ordering in $(B, +, \cdot)$. If there exists in B a minimal element

and a maximal element which are comparable then there exists a minimal element β and a maximal α in B such that $\beta \leqq 1 \leqq \alpha$.

<u>Proof</u>. Suppose each of m and M is, respectively, a minimal and a maximal element in B with $m \leqq M$. Let $x \in B$ such that $1 \leqq M + m^* \leqq x$. Then $x = (M + m^*)x = Mx + m^*x$. Now, from the maximality of M and $1 \leqq x$, we have that $M = Mx$. Also, $x^* \leqq 0$ and the minimality of m implies that $m + x^* = m$ and thus $m^*x = m^*$. Hence $x = Mx + m^*x = M + m^*$ and consequently $\alpha = M + m^*$ is maximal. In a similar manner $\beta = m + M^* \leqq 1$ is minimal.

<u>Theorem 1</u>. Suppose $\leqq$ is a compatible partial ordering in $(B, +, \cdot)$ which possesses a comparable pair of minimal and maximal elements. Let each of β and α (see Lemmas 1 and 2) denote respectively the minimal and maximal element in B such that $\beta \leqq 1 \leqq \alpha$. For each m in B , with $\alpha\beta + m = 1$, let $T_m = \{x \mid x \in B$ and $\beta m \leqq x \leqq \alpha m\}$. Then

(i) $(T_1, \leqq)$ is a Boolean lattice with $x \vee y = (x + y)\alpha + xy$ and $x \wedge y = (x + y)\beta + xy$ for each x and y in T_1 ,

(ii) $(T_m, \leqq)$ is isomorphic to $(T_n, \leqq)$ and no element in T_m is comparable with an element in T_m for $m \neq n$,

(iii) each element of B is a member of some T_m .

<u>Proof</u>. Let $x, y \in T_1$. Then $(x + y)\alpha + xy \in T_1$

since $\beta \le x$, $y \le \alpha$ implies that $\beta \le \alpha\beta \le (x+y)\alpha \le \alpha$ and $\beta \le xy < \alpha$ and thus the inequality $\beta \le (x+y)\alpha + xy \le \alpha$. The element $(x+y)\alpha + xy$ is an upper bound of x and y since $x = x(x+y) \le (x+y)\alpha$ and $y = y(x+y) \le (x+y)\alpha$ implies that $x = x + xy \le (x+y)\alpha + xy$ and $y = y + xy \le (x+y)\alpha + xy$. Now suppose $x \le \mu$ and $y \le \mu$ for some $\mu \in T_1$. Then $x + y \le \mu$ and $xy \le \mu$ and thus $(x+y)\alpha + xy \le \mu\alpha + \mu = \mu$. Hence $x \vee y = (x+y)\alpha + xy$ is the least upper bound of x and y. In a similar manner, $x \wedge y = (x+y)\beta + xy$ is in T_1 and is the greatest lower bound of x and y. Therefore $(T_1, \le)$ is a lattice.

Next, it is readily verified that $\alpha x + \beta x^* + \alpha\beta \in T_1$ for each $x \in B$. Now, let τ denote the mapping from B into T_1 such that $\tau(x) = \alpha x + \beta x^* + \alpha\beta$ for each $x \in B$. Then, by direct computation, it is seen that τ is a lattice - homomorphism from the Boolean lattice $(B, R_1(\le))$ into $(T_1, \le)$. In addition, the mapping τ is onto T_1 since, for each $y \in T_1$, $\tau(\alpha y + \beta y^* + \alpha\beta) = y$. Hence $(T_1, \le)$ is a homomorphic image of $(B, R_1(\le))$ and thus $(T_1, \le)$ is a Boolean lattice.

For part (ii), consider the mapping $\varphi: T_1 \to T_m$ where $\varphi(x) = mx$. The mapping φ is subjective since for each y in T_m, $\alpha\beta + y \in T_1$ and $\varphi(\alpha\beta + y) = \alpha\beta m + my = \alpha\beta y + my = (\alpha\beta + m)y = y$. Now suppose $\varphi(x_1) \le \varphi(x_2)$. Then $\alpha\beta + \varphi(x_1) \le \alpha\beta + \varphi(x_2)$ implies that $\alpha\beta + x_1 \le \alpha\beta + x_2$. Since $(\alpha\beta)^* + x_1 = 1 = (\alpha\beta)^* + x_2$, we thus obtain from $(\alpha\beta + x_1)((\alpha\beta)^* + x_1) \le (\alpha\beta + x_2))((\alpha\beta)^* + x_2)$ that $x_1 \le x_2$.

Similarly, $x_1 = x_2$ if $\varphi(x_1) = \varphi(x_2)$. Hence φ is an order isomorphism since, in addition, $x_1 \leq x_2$ obviously implies that $\varphi(x_1) \leq \varphi(x_2)$. Therefore $(T_1, \leq)$ is isomorphic to $(T_m, \leq)$.

To complete part (ii), let $y \in T_m$, $x \in T_n$ and suppose $y \leq x$. Now $\beta m \leq y \leq \alpha m$ and $\beta n \leq x \leq \alpha n$ implies that $\alpha\beta m = \alpha\beta y$ and $\alpha\beta n = \alpha\beta x$. Thus $\alpha\beta m \leq \alpha\beta n$ since $y \leq x$. Hence

$$m \leq \alpha\beta n + m = (\alpha\beta + m)(n+m) = n+m = (\alpha\beta + n)(m+n) = \alpha\beta m + n \leq m .$$

Therefore $m + n^* \leq 1$ and $1 \leq n + m^*$. Now from $m + n^* \leq 1$ and the minimality of β we obtain $\beta(m + n^*) = \beta$ and thus $\beta \leq m + n^*$ since $\beta \leq 1$ implies that $\beta(m+n^*) \leq m + n^*$. Consequently $\beta n \leq mn$. In a similar manner, $1 \leq n + m^*$ and the maximality of α implies that $n + m^* \leq \alpha$ and thus $mn \leq \alpha n$. Therefore $\beta m \leq m \leq mn \leq \alpha m$ and $\beta n \leq mn \leq n \leq \alpha n$. Thus $\alpha\beta m = (\alpha\beta)(mn) = \alpha\beta n$. Hence, from $(\alpha\beta)m = (\alpha\beta)n$ and $\alpha\beta + m = 1 = \alpha\beta + n$, $m = n$. Therefore, if $m \neq n$, no element of $(T_m, \leq)$ is comparable to an element of $(T_n, \leq)$.

Part (iii) follows from the fact that if $x \in B$ then $x \in T_m$ for $m = x + (\alpha\beta)^*$.

<u>Corollary</u>. Suppose $\leq$ is a compatible partial ordering in $(B, +, \cdot)$. A necessary and sufficient condition that $(B, \leq)$ be a Boolean lattice is that there exists an element $\alpha \in B$ such that $1 \leq \alpha$ and $0 \leq \alpha$.

<u>Proof</u>. Let $x \in B$. Now $1 \leq \alpha$ and $0 \leq \alpha$ implies that $x \leq \alpha x$ and $x \leq \alpha + x$. From $x \leq \alpha x$ we

obtain $\alpha + x \leq \alpha$ and thus $x \leq \alpha$. In a similar manner, $\alpha^* \leq 0$ and $\alpha^* \leq 1$ implies that $\alpha^* \leq x$. Hence $\alpha^* \leq$ $\leq x \leq \alpha$ and thus, by Theorem 1, $(B, \leq) = (T_1, \leq)$. Since the condition is obviously necessary, this completes the proof.

<u>Theorem 2</u>. Suppose $\leq$ is a compatible partial ordering in $(B, +, \cdot)$. A necessary and sufficient condition for every such poset $(B, \leq)$ to be isomorphic to a cardinal sum of isomorphic Boolean lattices is that $(B, +, \cdot)$ be finite.

<u>Proof</u>. If $(B, +, \cdot)$ is finite then $\leq$ obviously possesses a pair of comparable minimal and maximal elements and thus, by Theorem 1, $(B, \leq)$ is isomorphic to a cardinal sum of isomorphic Boolean lattices.

Now suppose every such $(B, \leq)$ is isomorphic to cardinal sum of isomorphic Boolean lattices. We first want to show that $(B, +, \cdot)$ is complete and atomic and thus be able to conclude that $(B, +, \cdot)$ is isomorphic to the algebra of all subsets of some set. Let K denote a non-empty subset of B and let $D = \{\sum_{i=1}^{n} k_i b_i \mid k_i \in K,\ b_i \in B$, and $n \in \mathfrak{z}^+\}$. To simplify notation, let $\leq_1$ denote the natural partial ordering $R_1(\leq)$. For each $p, q \in B$, define $p \leq q$ if and only if there exist $t \in B$, $d_i, d_j \in D$ such that $p = d_i + t$, $q = d_j + t$ and $p \leq_1 q$. Then $\leq$ is a compatible partial ordering of $(B, +, \cdot)$ and no element not in D is comparable with an element of D . Thus $(B, \leq)$ is the cardinal sum of isomorphic Boolean lattices L_i and consequently $D = L_i$ for some i. Hence the largest element u of D is, from the definition of D, the least upper bound of the members of X. Thus X has a least upper bound, with respect to the natural partial ordering $\leq_1$, and consequently $(B, +, \cdot)$ is complete. Next suppose $(B, +, \cdot)$ is not atomic. Then there exists a $b \in B$, $b \neq 0$, such that if $a \neq 0$ and $a \leq_1 b$ then a is not an atom of B. Hence, by the Hausdorff maximality principle, there exists in B, with respect to $\leq_1$, a maximal descending chain C from $b, \ldots \leq_1 a_\gamma \leq_1 \ldots \leq_1 a_\delta = b$, such that each $a_i \neq 0$ and $\bigwedge_i a_i = 0$. Let $H = \{h_i \mid h_i \in B$ and $a_\gamma \leq_1 h_i$ for some a_γ of the chain $C\}$. For each p, $q \in B$, define $p \leq q$ if and only if there exist s, $t \in B$ and an h_i, $h_j \in H$ such that $p = h_i s + t$, $q = h_j s + t$, and $p \leq_1 q$. Again, $\leq$ is a compatible partial ordering of $(B, +, \cdot)$ and no element not in H is comparable with an element of H. Thus, from compatibility and the hypothesis on each such poset $(B, \leq)$, there exists a Boolean lattice L_j in $(B, \leq)$ such that $H = L_j$. Hence $(B, +, \cdot)$ is atomic since $\bigwedge_i a_i = 0$ contradicts the fact that the smallest element x of H has the property that $x \neq 0$. Therefore $(B, +, \cdot)$ is isomorphic to the algebra of all subsets of some set U. Thus, without loss of generality, we may assume that $B = P(U)$ and that $+$ and $\cdot$ denote, respectively, set union and set intersection. Finally we want to show that $(B, +, \cdot)$ is finite. Assume that U is an infinite set. Let M denote the collection of all members of B which are finite subsets of $\mathcal{U}$. Define $p \leq q$ if and only if there exist a $t \in B$ and an $m_i, m_j \in \mathcal{M}$ such that $p = m_i + t$, $q = m_j + t$, and $p \leq_1 q$. Again we can conclude that $\leq$ is a compatible partial ordering of $(B, +, \cdot)$ and that no element not in $\mathcal{M}$ is comparable with an element of $\mathcal{M}$. Thus, from compatibility and the hypothesis on $(B, \leq)$ there exists a Boolean lattice L_k in $(B, \leq)$ such that $\mathcal{M} = L_k$. Hence some finite subset of $\mathcal{U}$ is the largest element in $\mathcal{M}$ and this is not possible since $\leq$ coincides with $\leq_1$ in $\mathcal{M}$ and $\mathcal{U}$ is infinite. Therefore $(B, +, \cdot)$ is finite.

<u>Theorem 3</u>. If $(B, +, \cdot)$ is a finite Boolean algebra of cardinality 2^n then the number of compatible partial orderings in $(B, +, \cdot)$ is 3^n.

<u>Proof</u>. For each $(\alpha, \beta) \in B \times B$ with $\alpha + \beta = 1$, $R(\leq) = \{(x,y) \mid (x+y)\beta + xy = x$ and $(x+y)\alpha + xy = y\}$ is a compatible partial ordering in B such that α is maximal, β is minimal, and $\beta \leq 1 \leq \alpha$. Hence by Theorem 1 and the preceding statement, the enumeration of the compatible partial orderings in $(B, +, \cdot)$ is reduced to determining the cardinality of the set $S = \{(x,y) \mid (x,y) \in B \times B$ and $x + y = 1\}$. The task of counting the number of elements in S is accomplished by referring to the associated Boolean lattice $(B, R_1(\leq))$ of $(B, +, \cdot)$. For each element x of dimension k, $0 \leq k \leq n$, in $(B, R_1(\leq))$ there are 2^k elements y in B such that $x + y = 1$. Thus, since there are $C_{n,k}$ elements in $(B, R_1(\leq))$ which have dimension k,

there are $2^k C_{n,k}$ ordered pairs $(x,y) \in B \times B$ such that x has dimension k and $x + y = 1$. Hence the cardinality of S is $\sum_{k=0}^{n} 2^k C_{n,k} = 3^n$.

R e f e r e n c e s

[1] G. BIRKHOFF: Lattice Theory, American Mathematical Society, Providence, R.I., 1967.

North Carolina State University
Department of Mathematics
Raleigh, N.C. 27607
U.S.A.

(Oblatum 1.3.1973)