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A REMARK ON n-TORSION-FREE MODULES

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Abstract:

D.R. Stone [7] has studied the n -purities and related notions. Among other results he showed the existence of a ring having a torsion-free but not 2-torsion-free (left) module. In this note we shall extend this result to arbitrary n . So, the purpose of this paper is to prove the following theorem:
For any natural integer n there exists a ring R and a (left) R -module M which is n-torsion-free but not $(n+1)$ -torsion-free.

Key-words:

purity, torsion-free module, Γ -flat, n -fir.

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1. Introduction. The essential step of the proof uses an example of S. Jondrup [5].

R will always denote a ring with identity; all modules will be unitary and left modules. R_n stands for the full matrix ring of degree n over R and for a module A , A^n means the n-th power of A, $A^n = A \oplus A \oplus \dots \oplus A$. It will be convenient to consider the elements of A^n as the $(n,1)$ -matrices (i.e. the column vectors).

2. Purities. Let Γ be a class of couples (F, U) where U is a submodule of a free module F. We say that a monomorphism $A \xrightarrow{i} B$ is Γ-pure if for any commutative diagram

$$(*) \qquad \begin{array}{ccc} U & \xrightarrow{\chi} & F \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{i} & B \end{array}$$

with $(P, U) \in \Gamma$, χ the canonical embedding, there exists a homomorphism $\psi : F \longrightarrow A$ with $\psi\chi = \varphi$ (see [6]). Let Γ_n be the class of all couples (F, U) where U is a submodule of a free module F and both F and U can be generated by n elements. D.R. Stone [7] has called a monomorphism $A \xrightarrow{i} B$ n-pure if the induced monomorphism $A^n \longrightarrow B^n$ is Γ_1-pure over R_n (see also (1.52) in [6]).

2.1. Proposition. A monomorphism $A \xrightarrow{i} B$ is n-pure iff it is Γ_n-pure.

Proof. We shall consider the diagrams $(*)$ and

$$(**) \qquad \begin{array}{ccc} R_n A & \xrightarrow{\chi'} & R_n \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle h'} \\ A^n & \xrightarrow{i'} & B^n \end{array}$$

where $A \in R_n$, χ' is the canonical embedding and i' is induced by i.

First suppose i is n-pure and let $(*)$ be a commutative diagram with $(F, \mathcal{U}) \in \Gamma_n$. Let $x_1, x_2, \dots, x_m$ be the free generators of F and $u_1, u_2, \dots, u_m$ the generators of $\mathcal{U}$ (it is easy to see that in the following there is no loss of generality in the assumption F has exactly m free generators; some of u_i's can be zeros, in general). Writing $u_i = \sum_{j=1}^{m} \alpha_{ij} x_j$, $i = 1, 2, \dots, m$, we put $\mathbb{A} = (\alpha_{ij})$, $\varphi'(A) = (\varphi(u_1), \varphi(u_2), \dots, \varphi(u_m))$ and $h'(\mathbb{I}) = (h(x_1), h(x_2), \dots, h(x_m))$ where $\mathbb{I}$ is the identity of R_m . φ' and h' induce the homomorphisms $\varphi' : R_m \mathbb{A} \longrightarrow A^n$, $h' : R_m \longrightarrow B^n$ since for $\mathbb{C} = (c_{ij})$, $\mathbb{C}\mathbb{A} = 0$ we have

$$\mathbb{C} \cdot \varphi'(\mathbb{A}) \overset{1)}{=} \sum_{j=1}^{m}(c_{1j}\varphi(u_j), \dots, \sum_{j=1}^{m}(c_{nj}\varphi(u_j)) = (\sum_{j=1}^{m} c_{1j} h(\sum_{i=1}^{m} \alpha_{ji} x_i), \dots$$

$$\dots, \sum_{j=1}^{m} c_{uj} h(\sum_{i=1}^{m} \alpha_{ji} x_i)) = \mathbb{C}\mathbb{A} \cdot h'(\mathbb{I}) = 0 .$$

Further

$$h'(A) = \mathbb{A} \cdot h'(\mathbb{I}) = \mathbb{A} \cdot (h(x_1), \dots, h(x_m)) =$$

$$= (\sum_{j=1}^{m} \alpha_{1j} h(x_j), \dots, \sum_{j=1}^{m} \alpha_{uj} h(x_j)) = (h(u_1), \dots$$

$$\dots, h(u_m)) = (\varphi(u_1), \dots, \varphi(u_m)) = \varphi'(\mathbb{A})$$

and $(**)$ commutes. By hypothesis there exists $\psi' : R_m \longrightarrow A^m$ with $\psi'\chi' = \varphi'$. If $(a_1, a_2, \dots, a_m)$ is the image of $\mathbb{I}$ under ψ' then we define $\psi : F \longrightarrow A$ by $\psi(x_i) = a_i$, $i = 1, 2, \dots, m$. It is easy to see that $\psi\chi = \varphi$ and

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1) Here and in the following $\varphi'(A)$ is assumed to be a $(1, m)$-matrix.

hence i is Γ_n -pure.

Conversely suppose i is Γ_n -pure and let $(**)$ be a commutative diagram with $\mathbb{A} = (\alpha_{ij})$, $\varphi'(\mathbb{A}) = (a_1, a_2, \ldots$ $\ldots, a_m)$, $h'(\mathbb{I}) = (b_1, b_2, \ldots, b_n)$. It follows from the commutativity of $(**)$ that $a_i = \sum_{j=1}^{n} \alpha_{ij} b_j$, $i = 1, 2, \ldots, m$. Let F be a free module with $x_1, x_2, \ldots, x_n$ as free generators and let U be a submodule of F generated by $u_i = \sum_{j=1}^{n} \alpha_{ij} x_j$, $i = 1, 2, \ldots, m$. From $\sum_{i=1}^{m} \lambda_i u_i = 0$ it follows $\sum_{i=1}^{n} \lambda_i u_i = \sum_{j=1}^{n} (\sum_{i=1}^{m} \lambda_i \alpha_{ij}) x_j = 0$ hence $\sum_{i=1}^{n} \lambda_i \alpha_{ij} = 0$, $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \lambda_i a_i =$

$$= \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \alpha_{ij} b_j = \sum_{j=1}^{n} (\sum_{i=1}^{m} \lambda_i \alpha_{ij}) b_j = 0 .$$

Therefore the map $\varphi(u_i) = a_i$, $i = 1, 2, \ldots, m$, induces the homomorphism $\varphi : U \longrightarrow A$. Defining $h : F \longrightarrow A$ by $h(x_i) = b_i$, $i = 1, 2, \ldots, m$, one can easily verify the commutativity of $(*)$. By hypothesis there exists $\psi :$ $: F \longrightarrow A$ with $\psi\chi = \varphi$. It is easy to see that for $\psi' : R_m \longrightarrow A^n$ defined by $\psi'(\mathbb{I}) = (\psi(x_1), \ldots, \psi(x_m))$ there is $\psi'\chi' = \varphi'$ and therefore the proof is finished.

3. <u>Flatness</u>. Following [6] we shall say that a module E is Γ_n -flat if for any short exact sequence $0 \longrightarrow A \xrightarrow{i}$ $\xrightarrow{i} B \longrightarrow E \longrightarrow 0$ the monomorphism i is Γ_n -pure.

Owing to [7], Prop. 3.2 we can say that a module E is torsion-free if it is Γ_1 -flat and it is n -torsion-free if E^n is torsion-free over R_n. By Proposition 2.1 and [7], Prop. 3.3, we have:

3.1. <u>Proposition</u>. A module E is n -torsion-free iff it is Γ_n -flat.

For completeness we shall introduce the following:

3.2. Lemma. A module E is Γ_n -flat iff there exists a short exact sequence $0 \to A \xrightarrow{i} F \to E \to 0$ with E free and i Γ_n -pure.

Proof. See (1.12) in [6].

Recall that a ring R is called (left) n -fir (n - free ideal ring) if any left ideal of R generated by n elements is a free module of uniquely determined rank (see [4]).

3.3. Lemma. Any left ideal of an n -fir is Γ_n -flat.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\chi} & F & & & & \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle h} & & & & \\ 0 & \longrightarrow & U' & \xrightarrow{\iota} & F' & \xrightarrow{\sigma} & I & \longrightarrow & 0 \end{array}$$

with exact rows, $(F, U) \in \Gamma_n$, F free and I a left ideal of R . This diagram induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\chi} & F & & & & \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle h} & & & & \\ 0 & \longrightarrow & U' & \xrightarrow{\iota} & \{\mathrm{Im}\, h, \mathrm{Im}\, \iota\} & \xrightarrow{\sigma} & I' & \longrightarrow & 0 \end{array}$$

with exact rows where $I' \subseteq I$ is a left ideal of R having n generaters. Hence the second row splits by some $\pi : \{\mathrm{Im}\, h, \mathrm{Im}\, \iota\} \longrightarrow U'$ (since R is an n -fir). For $\psi = \pi h : F \longrightarrow U'$ we have $\psi\chi = \pi h \chi = \pi \iota \varphi = \varphi$

and it suffices to use Lemma 3.2.

4. The proof of Theorem. Let n be a natural integer and let R be the K-algebra (K is a commutative field) on the $2(n+1)$ generators X_i, Y_i, $i = 1, 2, \ldots, n+1$, and defining relation $\sum_{i=1}^{n+1} X_i Y_i = 0$. There is shown in [5] that R is an n-fir and the left ideal I of R generated by $Y_1, Y_2, \ldots, Y_{n+1}$ is not flat. It remains only to show that I is not Γ_{n+1}-flat since it is Γ_n-flat by Lemma 3.3.

Let $0 \longrightarrow K \xrightarrow{i} F \xrightarrow{\sigma} I \longrightarrow 0$ be a short exact sequence where F is free with $Z_1, Z_2, \ldots, Z_{n+1}$ as free generators, σ is defined by $\sigma(Z_i) = Y_i$, $i = 1, 2, \ldots$ $\ldots, n+1$, and i is the canonical embedding of $K = \mathrm{Ker}\,\sigma$ into F. It is not too hard to derive from the definition of R that K is generated by $\sum_{i=1}^{n+1} X_i Z_i$. Therefore i is not Γ_{n+1}-pure since the converse would lead to the projectivity and hence to the flatness of I.

4.1. Corollary. For any natural integer n there exist a ring R and an R-module monomorphism which is n-pure but not $(n+1)$-pure.

Proof. The monomorphism i from the above proof has the desired property.

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