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LOCAL ERGODIC PROPERTIES OF L_p-OPERATOR SEMIGROUPS

Ryotaro SATO , Sakado

Abstract: In this note, utilizing a method of T.R. Terrell [The local ergodic theorem and semigroups of nonpositive operators, J.Functional Analysis 10(1972),424-429], a necessary and sufficient condition is given for a semigroup $\Gamma = \{T_t : t \geqq 0\}$ of bounded linear operators in an L_p-space with $1 \leqq p < \infty$ which is strongly integrable over every finite interval and of type C_1 to satisfy the local ergodic theorem.

Key words: Local ergodic theorem, L_p-operator semigroup, strong integrability, strong continuity.

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The main result

Let $(X, \mathcal{F}, m)$ be a σ-finite measure space and $1 \leqq p < \infty$. Let $\Gamma = \{T_t : t \geqq 0\}$ be a semigroup of bounded linear operators in $L_p = L_p(X, \mathcal{F}, m)$, i.e. $T_0 = I$ (the identity operator), $T_{s+t} = T_t T_s$, and $\|T_t\|_p < \infty$. In this section we shall assume that Γ satisfies the following two conditions:

(α) For any $f \in L_p$, $T_t f$ is integrable with respect to Lebesgue measure on every finite interval

$[a, b] \subset [0, \infty)$.

(β) For any $f \in L_p$, strong-$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h T_t f\, dt = f$.

It follows ([1], p.686) that for each $f \in L_p$ there exists a scalar function $T_t f(x)$, measurable with respect to the product of Lebesgue measure and m, such that for almost all t, $T_t f(x)$ belongs, as a function of x, to the equivalence class of $T_t f$. Moreover there exists a set $N(f) \in \mathcal{F}$ with $m(N(f)) = 0$, dependent on f but independent of t, such that if $x \notin N(f)$, then $T_t f(x)$ is integrable on every finite interval $[a, b]$ and the integral $\int_a^b T_t f(x)\, dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t f\, dt$. From now on we shall write $S_a^b f(x)$ for $\int_a^b T_t f(x)\, dt$.

<u>Theorem 1</u>. The following two conditions are equivalent:

(i) For any $f \in L_p$, $\lim_{b \downarrow 0} \frac{1}{b} S_0^b f(x) = f(x)$ a.e.

(ii) There exists a constant $c > 0$ such that for any $f \in L_p$ and any $\delta > 0$,

$$m(\{x ; \limsup_{b \downarrow 0} \frac{1}{b} S_0^b f(x) > \delta\}) \leq \frac{c}{\delta^p} \int |f|^p dm .$$

<u>Proof</u>. We proceed as in [2]. (i) $\Longrightarrow$ (ii): If $f \in L_p$ and $\delta > 0$, then

$$m(\{\limsup_{b \downarrow 0} \frac{1}{b} S_0^b f > \delta\}) = m(\{f > \delta\}) \leq \frac{1}{\delta^p} \int |f|^p dm .$$

(ii) $\Longrightarrow$ (i): Suppose that (ii) holds but (i) does not. Then there exists an $f \in L_p$ with

$$m(\{\limsup_{b \downarrow 0} \frac{1}{b} S_0^b f > f\}) > 0 .$$

Choose an $A \in \mathcal{F}$ with $0 < m(A) < \infty$ and $A \subset$ $\subset \{\limsup_{b \downarrow 0} \frac{1}{b} S_0^b f > f\}$, and let $a > 0$ be such that

(1) $$m(A \cap \{\limsup_{b \downarrow 0} \frac{1}{b} S_0^b f > f + a\}) = d > 0 .$$

Since strong-$\lim_{b \downarrow 0} \frac{1}{b} S_0^b f = f$ by (β), there exists an $f_0 \in L_p$ such that

$$\int |f - f_0|^p \, dm < \min\left(\frac{a^p d}{2^{p+2}}, \frac{a^p d}{2^{p+1} c}\right)$$

and

$$\lim_{b \downarrow 0} \frac{1}{b} S_0^b f_0(x) = f_0(x) \qquad \text{a.e.}$$

It follows that

$$m(\{a + f - f_0 \leqq \frac{a}{2}\}) \leqq m(\{|f - f_0| \geqq \frac{a}{2}\})$$
$$\leqq \left(\frac{2}{a}\right)^p \int |f - f_0|^p \, dm < \frac{d}{4} .$$

On the other hand (1) implies that

$$m(\{\limsup_{b \downarrow 0} \frac{1}{b} S_0^b (f - f_0) > a + f - f_0\})$$
$$= m(\{\limsup_{b \downarrow 0} \frac{1}{b} S_0^b f > a + f\}) \geqq d .$$

Thus we have

$$m(\{\limsup_{b \downarrow 0} \frac{1}{b} S_0^b (f - f_0) > \frac{a}{2}\})$$
$$\geqq \frac{3d}{4} > \frac{d}{2} > \left(\frac{2}{a}\right)^p c \int |f - f_0|^p \, dm ,$$

a contradiction. This completes the proof.

An application

In this section we shall assume that $\Gamma = \{T_t ; t \geq 0\}$ is a strongly continuous semigroup of linear contractions in L_1, i.e., $\|T_t\|_1 \leq 1$ for any $t \geq 0$, and the mapping $t \longrightarrow T_t f$ is continuous in the strong topology for any $f \in L_1$. Suppose, in addition, that there exists a constant $K > 0$ such that $\|T_t f\|_\infty \leq K \|f\|_\infty$ for any $f \in L_1 \cap L_\infty$. By the Riesz convexity theorem Γ may be considered to be a strongly continuous semigroup of bounded linear operators in L_p for each p with $1 \leq p < \infty$.

Theorem 2. For any $f \in L_p$ with $1 \leq p < \infty$,

$$\lim_{b \downarrow 0} \frac{1}{b} S_0^b f(x) = f(x) \qquad \text{a.e.}$$

Proof. In the case of $p = 1$, the theorem is proved by Terrell [2]. Hence we will consider here only the case of $1 < p < \infty$. As in [1, VIII.7], for $f \in L_p$ and $a > 0$, let

$$f^* = \sup_{0 < b < \infty} \left| \frac{1}{b} S_0^b f \right|, \quad e(a) = \{x ; |f(x)| > a\}$$

and

$$e^*(a) = \{x ; f^*(x) > a\} .$$

Then it follows easily from arguments analogous to those given in [1, VIII.7] that

$$a\,m(e^*(2Ka)) \leq \int_{e(a)} |f|\,dm$$

and

$$\int f^{*p}\,dm \leq \frac{p}{p-1}(2K)^p \int |f|^p\,dm \;.$$

Therefore for any $\delta > 0$,

$$m(\{\limsup_{b \downarrow 0} \frac{1}{b} S_0^b f > \delta\}) \leq m(\{\sup_{0<b<\infty} |\frac{1}{b} S_0^b f| > \delta\})$$
$$\leq \frac{1}{\delta^p}\int f^{*p}\,dm \leq \frac{1}{\delta^p}\left(\frac{p}{p-1}(2K)^p\right)\int |f|^p\,dm \;,$$

and hence Theorem 1 completes the proof.

Remark. Under the restriction that $K = 1$, the above theorem has been proved recently and independently by Mr. Y. Kubokawa. But his method of proof is quite different from ours.

R e f e r e n c e s

[1] DUNFORD N., SCHWARTZ J.T.: Linear Operators, Part I, Interscience,New York,1958.

[2] TERRELL T.R.: The local ergodic theorem and semigroups of nonpositive operators, J.Functional Analysis 10(1972),424-429.

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