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SMOOTHABILITY VERSUS DENTABILITY

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Abstract: A subset K of a Banach space X is smoothable if for every $\varepsilon > 0$ there is an $f \in X^*$, with $\sup\{f(u): u \in K\} = 1$, such that some closed ball B contains the set $\{u \in K: f(u) < 1-\varepsilon\}$ and $\sup\{f(u): u \in B\} < 1$. This notion is shown to have properties which parallel, in a sense, those possessed by dentability.

Key words: Smoothability, dentability, Fréchet differentiability.

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1. Introduction.

1.1. A subset K of a Banach space X will be said to be smoothable, if, for every $\varepsilon > 0$, there is an $f \in X^*$ with $\sup\{f(u): u \in K\} = 1$ such that some closed ball contains the set

$$K^{\varepsilon} = \{u \in K: f(u) \leq 1-\varepsilon\}$$

and is disjoint from the hyperplane $\{u: f(u) = 1\}$.

1.2. A subset K of a Banach space is called dentable if for every $\varepsilon > 0$, there is an $x \in K$ such that some hyperplane separates x from the set

$$K_\varepsilon = K \sim \overline{B(x, \varepsilon)}$$

where $B(x, \varepsilon)$ is the ball of radius ε about x.

The notion of dentability was introduced by Rieffel and used by him to prove a variant of the Radon-Nikodym theorem [6]. Among other things he proved that every bounded subset of $\ell^1(I)$, I an arbitrary set, is dentable and raised the question whether a dentable set must have a strongly exposed point. This question was answered in [2] where we showed that c_0 contains a closed and bounded convex body without any extreme points which nevertheless is dentable.

The definition of smoothability 1.1 parallels that of dentability in the sense that while in the latter K_ε is the complement of a closed ball and $x \in K$ is separated from K_ε by a hyperplane, in the former K^ε is the complement of a part cut off by a hyperplane and $f^{-1}[1]$ is separated from K^ε by a closed ball. It should be of interest to find out how the symmetry in definition is reflected in corresponding properties. It is the purpose of this note to bring out a number of parallels between smoothability and dentability. For example, the property of $\ell^1(I)$ with respect to dentability is duplicated by $c_0(I)$ with regard to smoothability; conversely ℓ^1 contains a nonsmoothable subset while c_0 contains a nondentable one. Finally, the term smoothability used here is strongly suggested by the fact that Fréchet differentiability of the norm implies smoothability of every bounded subset in the given space.

2. Smoothability properties of ℓ^1 and $m(I)$.

2.1. Proposition. The unit ball in ℓ^1 is not smoothable.

Proof. Let $0 < \varepsilon < \frac{1}{2}$ be given and suppose $f = (\lambda_1, \lambda_2, \ldots) \in m$ with $\|f\| = 1$ (i.e. $\sup|\lambda_i| = 1$). Suppose, for a contradiction, that a ball $B(x, r)$ centered at $x = (x_1, x_2, \ldots)$ and of radius r exists such that

$$(1) \qquad B^{\varepsilon} \subset \overline{B(x, r)}$$

where $B^{\varepsilon} = \{u : \|u\| \le 1,\ f(u) \le 1 - \varepsilon\}$, and

$$(2) \qquad \overline{B(x, r)} \cap f^{-1}[1] = \phi .$$

From (1) and (2) it clearly follows that, for some δ with $0 < \delta \le \varepsilon$,

$$(3) \qquad \sup\{f(u) : u \in B(x, r)\} = 1 - \delta .$$

Since B^{ε} is of diameter two, $r \ge 1$ is necessary to satisfy (1).
Hence, because of (2), $d = f(x) \le 0$ and

$$r = \inf\{\|x - u\| : u \in f^{-1}[1 - \delta]\} = 1 - d - \delta .$$

It now suffices to show that a y exists such that $\|y\| \le 1$, $f(y) \le \frac{1}{2}$ and $\|x - y\| > r$. Let then N be such that $\sum_{N+1}^{\infty} |x_i| < \frac{\delta}{2}$. If, for some $i \ge N + 1$ we have $\lambda_i = 0$ then $y = (y_1, y_2, \ldots)$ is chosen by setting $y_i = 1$ and $y_j = 0$ for $j \ne i$. Otherwise, choose any pair of indices i, j with $N + 1 \le i < j$

and set $y_i = \frac{1}{2} \frac{\lambda_i}{\|\lambda_i\|}$ and $y_j = -\frac{1}{2}\frac{\lambda_j}{\|\lambda_j\|}$ with $y_n = 0$ for other coordinates. In both cases $\|y\| = 1$ and, clearly, $f(y) < \frac{1}{2}$. Hence

$$\sum_{k=1}^{\infty} |x_k - y_k| = \|x\| + |y_i - x_i| + |y_j - x_j| - |x_i| - |x_j|$$

$$\geq \|x\| + 1 - 2(|x_i| + |x_j|) > \|x\| + 1 - \sigma .$$

Since $\|x\| \geq |f(x)| = -d$ we get $\|x - y\| > 1 - d - \sigma = r$.

2.2. Proposition. Every bounded subset K of $m(I)$, the space of all bounded real valued functions on the set I taken in the sup norm, is smoothable.

Proof. Let D be the diameter of K and choose a continuous linear functional f on $m(I)$ with $f(u) = u(i)$ for some $i \in I$ and all $u \in m(I)$. Since a translation and magnification has no effect on smoothability we may assume that $0 \in K$ and

$$\sup\{f(u) : u \in K\} = 1 .$$

Thus $|u(j)| \leq D$ for all $j \in I$ and all $u \in K$; and $u(i) \leq 1$ for all $u \in K$. Let $x \in m(I)$ be chosen so that $x(i) = -D$ and $x(j) = 0$ for $j \neq i$. If $\frac{1}{2} > \varepsilon > 0$ and $y \in K^{\varepsilon}$ then

$$\|x - y\| = \sup\{|x(j) - y(j)| : j \in I\} \leq D + 1 - \varepsilon .$$

Thus $K^{\varepsilon} \subset \overline{B(x, D + 1 - \varepsilon)}$ and, clearly $\overline{B(x, D+1-\varepsilon)}$ is disjoint from $f^{-1}[1]$.

2.3. As mentioned earlier, Rieffel showed in [6] that every subset of $\ell^1(I)$ is dentable and in [2] it was shown that the unit ball of m $(= m(\omega))$ fails to be dentable. Propositions 2.1 and 2.2 then exhibit the parallel behaviour of smoothability in that the unit ball of ℓ^1 $(= \ell^1(\omega))$ fails to be smoothable whereas all subsets of $m(I)$ are.

3. Conditions implying smoothability.

3.1. Proposition. Let X be a Banach space whose norm is Fréchet differentiable at some $x \in X$ with $\|x\| = 1$. Then every bounded subset of X is smoothable.

Proof. Suppose $K \subset X$ is bounded and $\varepsilon > 0$. Choose $f \in X^*$ such that $\|f\| = f(x) = 1$. We may assume that $\sup\{f(u): u \in K\} = 1$. Each ball of the form $B(-rx, r+1-\frac{\varepsilon}{2})$ with $r > 0$ is disjoint from $f^{-1}[1]$. A construction of Mazur [4, p. 131] shows that, for a suitably larger r, such a ball contains K^{ε}. Thus K is smoothable.

3.2. By a result of Asplund [1], a Banach space has a Fréchet differentiable norm on a set of second category if it has an equivalent norm whose dual norm is locally uniformly convex. All Banach spaces with a separable dual were shown by him to have this property. More recently, Troyanski [7] has shown that reflexive spaces too belong to the above class. Thus, the conclusion of the last proposition holds for all reflexive spaces and those having a separable dual.

3.3. As observed by Rieffel [6, p. 72] each compact subset of a Banach space is dentable. The next proposition shows that such sets also have the smoothability property. This is preceded by a simple lemma.

3.4. Lemma. Let Y be a closed subspace of a Banach space X. Suppose $K \subset Y$ is smoothable in Y. Then K

is smoothable (in $\mathfrak{X}$).

Proof. Let $\varepsilon > 0$ and suppose $f \in \mathfrak{Y}^*$ is such that $\sup \{f(u): u \in K\} = 1$. Suppose $B = \{y \in \mathfrak{Y}: \|x - y\| \leq r\} \supset K^{\varepsilon}$ for some $r > 0$ and $x \in \mathfrak{Y}$ with $B \cap f^{-1}[1] = \phi$. By the Hahn-Banach theorem, f extends to $\mathfrak{X}$ with preservation of norm and obviously, $\{u \in \mathfrak{X}: \|x - u\| \leq r\}$ separates K^{ε} from the closed hyperplane determined by f.

3.5. Proposition. If K is a compact subset of a Banach space $\mathfrak{X}$ then K is smoothable.

Proof. Let $\mathfrak{Y}$ by the closed subspace spanned by K. By the preceding lemma, it suffices to show that K is smoothable in $\mathfrak{Y}$. By a result of Mazur [5], the unit sphere of $\mathfrak{Y}$ contains a point y at which the norm is Gateaux differentiable. Let $f \in \mathfrak{Y}^*$ be such that $\|f\| = f(y) = 1$ and let $\varepsilon > 0$. We may clearly assume that $\sup\{f(u): u \in K\} = 1$ and, as in the proof of 3.1, we note that each member of the family $\{B(-ry, r + 1 - \frac{\varepsilon}{2}): r > 0\}$ is disjoint from $f^{-1}[1]$. As observed by Klee [3], the above family forms an open cover of the compact set K^{ε}. It readily follows that this last set is contained in some member of that family.

3.6. In closing we note that the class of Banach spaces consisting of all conjugate spaces whose dual is separable and of all reflexive spaces has the property that each bounded subset of any of its members is both smoothable and dentable. For smoothability this follows from the remarks made in 3.2. As for dentability any member of the above class is known to have the property that every bounded set in it has a strongly exposed point, and this last property is easily seen to imply dentability.

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References

[1] E. ASPLUND: Fréchet differentiability of convex functions, Acta Math.121(1968),31-47.

[2] M. EDELSTEIN: Concerning dentability, Pac.J.Math.(To appear.)

[3] V. KLEE: Extremal structure of convex sets, Math.Zeitschr. 69(1968),90-104.

[4] S. MAZUR: Über schwache Konvergenz in den Räumen (L^n), Studia Math.4(1933),128-133.

[5] S. MAZUR: Über konvexe Mengen in linearen normierten Räumen, Studia Math.4(1933),70-84.

[6] M.A. RIEFFEL: Dentable subsets of Banach spaces with applications to a Radon-Nikodym theorem, Proc. Conf.Functional Anal.,Thompson Book Co.Washington,1967,pp.71-77.

[7] S.L. TROYANSKI:On locally uniformly convex and differentiable norms in certain non-separable Banach spaces, Studia Math.37(1971),173-179.

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