Commentationes Mathematicae Universitatis Carolinae

Ivan Studnička

Non-existence of Cartesian groups of order $2p^m$

Commentationes Mathematicae Universitatis Carolinae, Vol. 13 (1972), No. 4, 721--725

Persistent URL: `http://dml.cz/dmlcz/105455`

Commentationes Mathematicae Universitatis Carolinae

13,4 (1972)

NON-EXISTENCE OF CARTESIAN GROUPS OF ORDER $2p^m$

Ivan STUDNIČKA, Brno

By a cartesian group there is meant an algebraic system $(S, +, \cdot, 0, 1)$ where $(S, +, 0)$ is an additive group (with a neutral element 0), $(S, \cdot)$ a multiplicative groupoid, further $0 \cdot x = x \cdot 0 = 0$, $1 \cdot x = x \cdot 1 = x$ hold for $\forall x \in S$ and finally the following "planarity" conditions are valid:

(A) $\forall r, s, t \in S,\ r \neq s \quad \exists!\, x \in S \quad x \cdot r = x \cdot s + t$,

(B) $\forall r, s, t \in S,\ r \neq s \quad \exists!\, y \in S \quad -r \cdot y = -s \cdot y + t$.

S termed the order of $(S, +, \cdot, 0, 1)$.

Cartesian groups are just Hall planar ternary rings of flag (A, a)-transitive planes provided a is the improper line and A the improper point of y-axis. Thus the existence or non-existence of any prescribed order n expresses the existence or non-existence flag (a, a)-transitive planes of order n . The purpose of the present note is to give a contribution to the open problem of finding all integers n for which there exists a flag (A, a)-transitive plane of order n .

AMS, Primary: 50D20 Ref. Ž. 3.915.4

The aim of this note is to prove that there is no cartesian group of order $2p^m$ (p odd prime, $m \geq 1$). In the sequel $(S, +, \cdot, 0, 1)$ will denote a fixed cartesian group of finite order $m+1$. Obviously $(S \setminus \{0\}, \cdot, 1)$ is a loop so that

(1) the multiplication table of it is a Latin square of order m.

First we affirm that

(2) for distinct rows $(x_1, \dots, x_m)$, $(y_1, \dots, y_m)$ of this Latin square the m differences $x_1 - y_1, \dots, x_m - y_m$ are always exactly all the elements of $S \setminus \{0\}$.

Proof. Suppose that the Latin square mentioned above is of the form

$$\begin{array}{c|cccc|}
 & a_1, & a_2, & \dots, & a_m \\
\hline
a_1 & a_1 a_1, & a_1 a_2, & \dots, & a_1 a_m \\
a_2 & a_2 a_1, & a_2 a_2, & \dots, & a_2 a_m \\
\vdots & & \vdots & & \\
a_m & a_m a_1, & a_m a_2, & \dots, & a_m a_m \\
\hline
\end{array}$$

where $\{a_1, \dots, a_m\} = S \setminus \{0\}$, $a_1 = 1$.

Thus the i-th and j-th rows are

$$(a_i \cdot a_1,\ a_i \cdot a_2, \dots, a_i \cdot a_m),$$

$$(a_j \cdot a_1,\ a_j \cdot a_2, \dots, a_j \cdot a_m)$$

and it suffices to show that $a_i \cdot a_k - a_j \cdot a_k \neq a_i \cdot a_l - a_j \cdot a_l$ $\forall i, j, k, l \in \{1, \dots, m\}$, $i \neq j$, $k \neq l$.

Suppose on the contrary $a_i \cdot a_k - a_j \cdot a_k = a_i \cdot a_l - a_j \cdot a_l$ for some $i, j, k, l \in \{1, \dots, n\}$, $i \neq j$, $k \neq l$. Putting $-a_i \cdot a_l + a_i \cdot a_k = a_j \cdot a_l + a_j \cdot a_k = t$, we get $a_i \cdot a_k = a_i \cdot a_l + t$, $a_j \cdot a_k = a_j \cdot a_l + t$. Now (A) implies $a_i = a_j$, a contradiction.

Now we shall use the elementary facts: G a group, G_0 its commutant $\Longrightarrow$ G/G_0 Abelian and G a group, H its normal subgroup, G/H Abelian $\Longrightarrow$ $G_0 \subseteq H$.

(3) Let G be a finite group (additive) with commutant G_0 such that $G_0, G_1, \dots, G_s$ are precisely all elements of G/G_0. If $K_1 + \dots + K_r \in G_j$ for some $j \in \{0, 1, \dots, s\}$ with $K_1, \dots, K_r \in G$ then each sum of $K_1, \dots, K_r$ independently of the order belongs to G_j.

In fact, for $i \in \{1, \dots, r\}$ there is a $s_i \in \{0, 1, \dots, s\}$ such that $K_i \in G_{s_i}$. Thus $G_{s_1} + \dots + G_{s_r} \subseteq G_j$. The required conclusion follows at once because G/G_0 must be Abelian.

(4) Let $(S, +, \cdot, 0, 1)$ be a finite cartesian group of order $n+1$. Then every sum $K_1 + \dots + K_n$ with pairwise distinct summands $K_1, \dots, K_n \in S \setminus \{0\}$ belongs to the commutant of the group $(S, +, 0)$.

<u>Proof.</u> Let $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ be distinct rows of the Latin square considered in (2). Thus first $\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\} = S \setminus \{0\}$. Further, by (2), $x_1 - y_1, \dots, x_n - y_n$ are pairwise distinct and non zero. Now remark that $x_1 - y_{i_1} + x_2 - y_{i_2} + \dots + x_n - y_{i_n} = 0$

(which belongs to the commutant of additive group considered) for $y_{i_1} = x_1, \dots, y_{i_n} = x_n$ so that, by (3), $x_1 - y_1 + x_2 - y_2 + \dots + x_n - y_n$ also belongs to the mentioned commutant. The members $x_1 - y_1, \dots, x_n - y_n$ are non-zero. Repeating use of (3) gives the conclusion of (4).

(5) If G is a group (additive) of even order and N its normal subgroup of odd order such that G/N is of order 2, then no sum of all elements of G belongs to G_0 (commutant of G).

Proof. Since G/N is Abelian it suffices to show that the sum considered does not belong to N. As G/N has order 2 (with elements $N = N_0, N_1$), it must be $N_1 + N_1 = N_0$. Let $K_1, \dots, K_{2k}$ be all the elements of G ($2k =$ order of G) denoted in such a manner that $K_1, \dots, K_k$, respectively $K_{k+1}, \dots, K_{2k}$ are all the elements of N_0 or of N_1, respectively. Then $K_1 + \dots + K_k \in N_0$, $K_{k+1} + \dots + K_{2k} \in N_1$ because $N_1 + N_1 = N_0$ and because N_1 has an odd order.

For the proof of our main result we need still two elementary facts about solvable groups, namely that no sum of all elements of any solvable finite group of even order belongs to its commutant and that every finite group with order $p^{\alpha} q^{\beta}$ (p, q prime, α, β integers) is necessarily solvable. The first assertion follows at once because every group of odd order $2m$ has at least one subgroup of order m and consequently of index 2. This subgroup must be normal and we can apply (5).

After these remarks, assertion (4) gives the non-existence of cartesian groups of order $2p^m$ (p odd prime, $m \geq 1$).

Using the Bruck-Ryser theorem we see that for odd m and $p \equiv 3 \pmod 4$ we get nothing new but the remaining cases seem to give a result which is not yet known.

R e f e r e n c e s

[1] M. HALL Jr.: The Theory of Groups, New York 1959.

[2] G. PICKERT: Projektive Ebenen, Berlin-Göttingen-Heidelberg 1955.

(Oblatum 16.10.1972)

Katedra matematiky FE-VUT
Hilleho 6, Brno
Československo