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TENSOR PRODUCTS IN THE CATEGORY OF CONVERGENCE SPACES

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Various additional structures on a category have been studied recently. One of them is the tensor product, whose non-trivial examples are likely to appear, sometimes in a great number, in categories of algebras or models of Gabriel theories. The aim of this paper is to discuss the existence of tensor products in the category of convergence spaces.

1. Convention. The class of all objects of a category $\mathcal{A}$ will be denoted by $|\mathcal{A}|$. Given $A, B \in |\mathcal{A}|$, we shall denote by $\langle A, B \rangle_{\mathcal{A}}$ the set of all morphisms from A to B in $\mathcal{A}$. Instead of " $f \in \langle A, B \rangle_{\mathcal{A}}$ " we shall also write " $A \xrightarrow{f} B \in \mathcal{A}$ ". The inverse of an isomorphism f will be denoted by $\overline{f}$.

By a concrete category we understand a couple $(\mathcal{A}, \mathcal{U})$ where $\mathcal{A}$ is a category and $\mathcal{U}$ is a faithful functor $\mathcal{A} \longrightarrow \mathcal{S}et$.

For $A, B \in |\mathcal{A}|$ we shall write " $A \leq B$ " whenever $\mathcal{U}A = \mathcal{U}B$ and the identity of the underlying set carries a morphism in $\mathcal{A}$. For two functors F, G in n variables

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with values in a concrete category, " $F \leq G$ " will stand for " $F(A_1,\dots,A_n) \leq G(A_1,\dots,A_n)$ for any objects $A_1,\dots,A_n \in |\mathcal{A}|$ " .

2. Definition. Let $(\mathcal{A},\mathcal{U})$ be a concrete category. A structure of the symmetric monoidal closed category (SMC) on $(\mathcal{A},\mathcal{U})$ consists of the following data:

(i) a functor $\otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ (tensor product),

(ii) an object $J \in |\mathcal{A}|$ together with natural equivalences

$$(A \otimes B) \otimes C \overset{a^{ABC}}{\approx} A \otimes (B \otimes C) ,$$

$$A \otimes B \overset{c^{AB}}{\approx} B \otimes A ,$$

$$A \otimes J \overset{r^{A}}{\approx} A ,$$

$$J \otimes A \overset{\ell^{A}}{\approx} A .$$

(iii) a strong hom-functor H in $(\mathcal{A},\mathcal{U})$, i.e. a functor $\mathcal{A}^* \times \mathcal{A} \longrightarrow \mathcal{A}$ linked with the ordinary hom-functor $\langle -,- \rangle_{\mathcal{A}}$ of $\mathcal{A}$ by means of the commutative diagram

$$\begin{tikzcd}
\mathcal{A}^* \times \mathcal{A} \arrow[rr, "H"] \arrow[dr, "\langle -,- \rangle_{\mathcal{A}}"'] & & \mathcal{A} \arrow[dl, "\mathcal{U}"] \\
& \mathcal{Set} &
\end{tikzcd}$$

(iv) an adjunction in two variables

$$\langle A \otimes B , C \rangle_{\mathcal{A}} \overset{n^{ABC}}{\approx} \langle A , H(B,C) \rangle_{\mathcal{A}} .$$

A couple of functors $\otimes$, H satisfying the conditions (i) and (iii) will be called a tensor couple in

$(\mathcal{A},\mathcal{U})$ provided that it can be completed to an SMC structure in $(\mathcal{A},\mathcal{U})$.

3. Remark. The natural isomorphisms sub (ii) alone do not enable us to treat $\otimes$ as an associative and commutative product with a bothsided unity J unless we show that they are coherent in the sense of [4]. Nevertheless, the fact that both tensor products we shall describe explicitly are coherent and that we shall dispense with the notion of coherence throughout the discussion of otherpossible cases allows us to omit it from our further consideration.

With this reservation, the essence of the above definition is taken from [2]. Let me recollect some of its immediate consequences.

1) The monoid $\langle J, J \rangle$ is commutative.

2) The identity functor of $\mathcal{A}$ is naturally equivalent with $H(J,-)$.

3) For any fixed C the contravariant functor $H(-,C)$: $: \mathcal{A}^* \longrightarrow \mathcal{A}$ is a right self-adjoint, therefore it transfers colimits to limits.

4) If $(\otimes, H)$, $(\otimes', H')$ are tensor couples in $(\mathcal{A},\mathcal{U})$, then for any natural transformation $\tau : \otimes' \longrightarrow \otimes$ there is exactly one natural transformation $\eta : H \longrightarrow H'$ making the diagram

$$\begin{tikzcd}
\langle A \otimes B, C \rangle \arrow[r, "\eta^{ABC}"] \arrow[d, "{\langle \tau^{AB}, 1_C \rangle}"'] & \langle A, H(B,C) \rangle \arrow[d, "{\langle 1_A, \eta^{BC} \rangle}"] \\
\langle A \otimes' B, C \rangle \arrow[r, "\eta'^{ABC}"] & \langle A, H'(B,C) \rangle
\end{tikzcd}$$

commutative for any $A, B, C \in |\mathcal{A}|$, and vice versa.

4. Example. Up to a natural equivalence, there is exactly one SMC structure in $(Set, 1_{Set})$, formed by the cartesian product $X \times Y$, the functor $\langle -,- \rangle_{Set}$, a one-point set $\{0\}$, and the isomorphisms $a_0^{XYZ}, c_0^{XY}, r_0^X, \ell_0^X$, p_0^{XYZ}, defined

$$a_0^{XYZ}((x,y),z) = (x,(y,z)) \quad ,$$

$$c_0^X(x,y) = (y,x) \; ,$$

$$r_0^X(x,0) = \ell_0^X(0,x) = x \; . \; x \in X, \; y \in Y, \; z \in Z \; ,$$

$$(p_0^{XYZ}\varphi(x))(y) = \varphi(x,y) ; \; X \times Y \xrightarrow{\varphi} Z \in Set ; \; x \in X, \; y \in Y$$

respectively.

5. Definition. Let ω denote the least infinite ordinal. By an L-space we shall understand a couple (X, Λ) where X is a set and $\Lambda \subset X^{\omega} \times X$ is a relation between sequences in X and elements of X satisfying the following axioms:

(L0) $(\{x_n\}, x^1), (\{x_n\}, x^2) \in \Lambda$ always implies $x^1 = x^2$.

(L1) If $x_n = x$ for all $n \in \omega$ then $(\{x_n\}, x) \in \Lambda$.

(L2) If $(\{x_n\}, x) \in \Lambda$ and $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$, then $(\{x_{k_n}\}, x) \in \Lambda$.

(L3) If $\{x_n\}, x$ are not in Λ, there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $(\{x_{k_{\ell_n}}\}, x) \in \Lambda$ does not hold for any subsequence $\{x_{k_{\ell_n}}\}$ of $\{x_{k_n}\}$.

The relation Λ will be called a convergence in X. We shall call an L-morphism any triple $((X,\Lambda),f,(X',\Lambda'))$ where (X,Λ), (X',Λ') are L-spaces and $X \xrightarrow{f} X'$ is a mapping such that $(\{x_m\},x) \in \Lambda$ always implies $(\{f(x_m)\},f(x)) \in \Lambda'$.

All L-spaces and L-morphisms form a category $\mathcal{L}$. We define an underlying set functor $U_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathbf{Set}$ by the rule

$$U_{\mathcal{L}}((X,\Lambda),f,(X',\Lambda')) = (X,f,X') .$$

Observe that owing to the axiom (L1) any constant mapping of the underlying sets carries a morphism in $\mathcal{L}$.

For the sake of brevity we shall in the sequel omit these part of the proofs that rest in plain verifying of some assumptions.

6. Proposition. Given a diagram $D : \Delta \longrightarrow \mathcal{L}$ in $\mathcal{L}$, $D(d) = (Xd, \Lambda d)$, $d \in |\Delta|$. Let $(X \xrightarrow{f_d} X_d)_{d \in |\Delta|}$ be the limit of $U_{\mathcal{L}} \circ D$ in $\mathbf{Set}$. Putting $\Lambda = \{(\{x_m\},x) \mid x_m, x \in X$ and for any $d \in |\Delta|$ we have $(\{f_d(x_m),f_d(x)) \in \Lambda d\}$, we obtain $((X,\Lambda) \xrightarrow{f} (X\ \ ,\Lambda\ ))_{d \in |\Delta|}$ as the limit of D in $\mathcal{L}$.

7. Proposition. $\varphi = ((X,\Lambda),f,(X',\Lambda'))$ is an epimorphism in $\mathcal{L}$ iff for any $x' \in X'$ there is a sequence $\{x'_m\}$ in $f(X)$ such that $(\{x'_m\},x') \in \Lambda'$.

Proof. Let M be the set of all the Λ'-limits of sequences in $f(X)$. Suppose $M \neq X'$. We shall use the usual push-out construction to prove that φ is not an epimorphism.

We put $X'' = M \cup ((X'-M) \times \{0\} \cup ((X'-M) \times \{1\})$ and define mappings $\mu : X'' \longrightarrow X'$, $\nu : X'' \longrightarrow \exp 2$ so that $\mu | M = 1_M$; $\mu(x', i) = x'$ for $x' \in X' - M$, $i = 0, 1$; $\nu(x') = \emptyset$ for $x' \in M$; $\nu(x', i) = \{i\}$ for $x' \in X' - M$, $i = 0, 1$. Then $\Lambda'' = \{(\{x''_n\}, x'') | x''_n, x'' \in X''; (\{\mu(x''_n)\}, \mu(x'')) \in \Lambda'$ and there is $m_0 \in \omega$ such that $\bigcap_{n \geq m_0} \nu(x''_n) \supset \nu(x'')\}$ is a convergence in X''. Putting $g_i | M = 1_M$ and $g_i(x') = (x', i)$ for $x' \in X' - M$, we obtain $\mathbf{L}$-morphisms $\gamma_i = ((X', \Lambda'), g_i, (X'', \Lambda'')), i = 0, 1$ such that $\gamma_0 \neq \gamma_1$ while $\gamma_0 \varphi = \gamma_1 \varphi$.

The other implication is a consequence of (LO).

In order to avoid the persistent occurence of the forgetful functor in the formulas and thus to make the text more readable, we shall in the further notation disregard the difference between an object of $\mathcal{L}$ and its underlying set.

We shall write "$x_n \longrightarrow x$ in X" instead of "$(\{x_n\}, x) \in \Lambda$" and say that $f : X \longrightarrow X'$ is continuous or that it is a morphism instead of "f carries a morphism in $\mathcal{L}$".

c_y^{XY} will denote the constant morphism mapping an $\mathbf{L}$-space X on $y \in Y$.

We shall use $\mathbf{A}$ to denote the set $\omega + 1$ of all ordinals at most equal to ω with the convergence of the ordered topological space $\mathbf{T}_{\omega+1}$.

8. Lemma. Let $\{x_n\}$ be a sequence of points in an $\mathbf{L}$-

space X . The following conditions are equivalent:

1) $x_n \longrightarrow x$ in X .

2) The mapping $A \xrightarrow{\varphi} X$ defined $\varphi(n) = x_n$ for $n \in \omega$, $\varphi(\omega) = x$ is continuous.

Proof. The non-trivial implication is $(1 \Longrightarrow 2)$. Let $x_n \longrightarrow x$ in X .

a) Assume that $a_n \longrightarrow m$ in A . Then from some n_0 on we have $a_n = m$ and every subsequence $\{\varphi(a_{k_n})\}$ of $\{\varphi(a_n)\}$ has $\{\varphi(a_{k_n})\}_{n \geq n_0}$ as a constant subsequence, which converges to $\varphi(m)$ in X . By (L3) $\varphi(a_n) \longrightarrow \varphi(m)$ in X .

b) Assume that $a_n \longrightarrow \omega$ in A . Then for any subsequence $\{\varphi(a_{k_n})\}$ of $\{\varphi(a_n)\}$ the sequence $\{a_{k_n}\}$ has a non-decreasing subsequence $\{a_{k_{\ell_n}}\}$. Either $a_{k_{\ell_n}} = \omega$ from some n_0 on and we proceed in the same manner as in a), or $\{a_{k_{\ell_n}}\}$ has an increasing subsequence $\{a_{k_{\ell_{p_n}}}\}$. In that case $\{\varphi(a_{k_{\ell_{p_n}}})\}$ is a subsequence of $\{x_n\}$ and converges to x in X . Again, $\varphi(a_n) \longrightarrow \varphi(\omega)$ in X. φ is continuous.

9. Proposition. The object A with all its endomorphisms in $\mathcal{L}$ forms a left adequate of $\mathcal{L}$ in the sense of [3].

Proof. Given any $X \in |\mathcal{L}|$ we define a small category $\mathcal{K}_{(A)}(X)$ with the set of objects $\langle A, X \rangle_{\mathcal{L}}$, whose morphisms are all the triples (f_1, α, f_2) where $f_1, f_2 \in \langle A, X \rangle_{\mathcal{L}}$, $\alpha \in \langle A, A \rangle_{\mathcal{L}}$ and $f_1 = f_2 \circ \alpha$.

The formula: $D_{(A)}(X)(f_1, \alpha, f_2) = \alpha$ defines a functor $\mathcal{K}_{(A)}(X) \rightarrow \mathcal{L}$. Putting $\eta^f = f$, $f \in |\mathcal{K}_{(A)}, (X)|$, we obtain a cocompatible family $\eta = (\eta^f)_{f \in |\mathcal{K}_{(A)}(X)|}$ of the diagram $\mathbf{D}_{(A)}(\mathcal{X})$. Our aim is to prove that η is the colimit of $\mathbf{D}_{(A)}(\mathcal{X})$ in $\mathcal{L}$.

Assume that $(A \xrightarrow{g^f} Y)_{f \in |\mathcal{K}_{(A)}(X)|}$ is another cocompatible family of $\mathbf{D}_{(A)}(\mathcal{X})$. We seek a continuous mapping $\mathcal{X} \xrightarrow{g} \mathcal{Y}$ such that $g \circ f = g^f$ for any $f \in |\mathcal{K}_{(A)}(\mathcal{X})|$. Thus for any $x \in \mathcal{X}$ it must satisfy the condition $g(x) = g^{c_x^{AX}}(\omega)$. (1)

On the other hand, using (1) as a definition of g we have for any $A \xrightarrow{f} x \in |\mathcal{K}_{(A)}(X)|$ and $a \in A$:

$g(f(a)) = g^{c_{f(a)}^{AX}}(\omega) = g^{f \circ c_a^{AA}}(a) = g^f \circ c_a^{AA}(\omega) = g^f(a)$.

It remains to prove that g is continuous. If $x_n \to x$ in $\mathcal{X}$ then the corresponding morphism φ from Lemma 8 is an object of $\mathcal{K}_{(A)}(\mathcal{X})$ and $g(x_n) = g \circ \varphi(n) = g^{\varphi}(n) \to g^{\varphi}(\omega) = g(x)$ since g^{φ} is continuous.

10. <u>Proposition</u>. Given $\mathcal{X}, \mathcal{Y} \in |\mathcal{L}|$, we shall denote by $\mathbf{B}(\mathcal{X}, \mathcal{Y})$ the set $\langle \mathcal{X}, \mathcal{Y} \rangle_{\mathcal{L}}$ with the (pointwise) convergence: $\xi_n \to \xi$ in $\mathbf{B}(\mathcal{X}, \mathcal{Y})$ iff $\xi_n(x) \to \xi(x)$ in $\mathcal{Y}$ for any $x \in \mathcal{X}$. Putting $\mathbf{B}(f, g) = \langle f, g \rangle$ for morphisms, we obtain a strong hom-functor in $(\mathcal{L}, \mathcal{U}_{\mathcal{L}})$.

11. <u>Proposition</u>. Given $\mathcal{X}, \mathcal{Y} \in |\mathcal{L}|$, we shall denote by $\mathcal{X} \oplus \mathcal{Y}$ the cartesian product of the underlying sets with the convergence: $(x_n, y_n) \to (x, y)$ in $\mathcal{X} \oplus \mathcal{Y}$ iff $x_n \to x$ in $\mathcal{X}$, $y_n \to y$ in $\mathcal{Y}$, and there is $n_0 \in \omega$ such that for all $n \geq n_0$ at least one of the

equalities $x_n = x$, $y_n = y$ holds. Completing the definition by $f \oplus g = f \times g$ for morphisms, we obtain a functor $\oplus : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$.

Proof. Let us verify the axiom (L3). If $(x_n, y_n) \not\to (x, y)$ in $X \oplus Y$, then either e.g. $x_n \not\to x$ in X , thus having a subsequence $\{x_{k_n}\}$ with no subsequence converging to x in X . Then we can claim the same for $\{(x_{k_n}, y_{k_n})\}, (x, y)$, and $X \oplus Y$. Or for an increasing sequence $\{\ell_n\}$ of natural numbers we have always $x_{\ell_n} \neq x$ and $y_{\ell_n} \neq y$. In that case $\{(x_{\ell_n}, y_{\ell_n})\}$ is a subsequence of $\{(x_n, y_n)\}$, no subsequence of which converges to (x, y) in $X \oplus Y$.

12. Proposition. $(\oplus, B)$ is a tensor couple in $(\mathcal{L}, U_{\mathcal{L}})$.

Proof. Denote by P the singleton in $\mathcal{L}$ - a one-point set $\{0\}$ with one convergent sequence. It is easy to verify that for any $X, Y, Z \in |\mathcal{L}|$ the bijections $a_0^{UX,UY,UZ}$, $c_0^{UX,UY}$, r_0^{UX} , ℓ_0^{UX} of the underlying sets carry isomorphisms in $\mathcal{L}$ and that the formula

$$\pi^{XYZ}\varphi(x)(y) = \varphi(x, y); \; X \oplus Y \xrightarrow{\varphi} Z \in \mathcal{L}, \, x \in X, \, y \in Y$$

defines correctly an adjunction $\oplus \dashv B$. Let us only clarify one thing that is not perhaps immediately seen. If $X \xrightarrow{\psi} B(Y, Z) \in \mathcal{L}$ and $(x_n, y_n) \longrightarrow (x, y)$ in $X \oplus Y$ then for any subsequence $\{\psi(x_{k_n})(y_{k_n})\}$ of $\{\psi(x_n)(y_n)\} = \{\bar{\pi}^{XYZ}\psi(x_n, y_n)\}$ there is an increasing sequence $\{\ell_n\}$

of natural numbers such that either $\{x_{k_{\ell_m}}\}$ or $\{y_{k_{\ell_m}}\}$ is constant. Hence $\bar{\pi}^{XYZ}\varphi(x_m, y_m) \to \bar{\pi}^{XYZ}(x,y)$ in Z .

13. Proposition. For $X, Y \in |\mathcal{L}|$ we shall denote by $D(X,Y)$ the set $\langle X, Y\rangle_{\mathcal{L}}$ with the (diagonal) convergence : $\xi_n \to \xi$ in $D(X,Y)$ iff $\xi_n(x_n) \to \xi(x)$ in Y whenever $x_n \to x$ in X . Defining $D(f,g) = \langle f, g\rangle$ for morphisms, we obtain a strong hom-functor in $(\mathcal{L}, U_{\mathcal{L}})$.

Proof. The non-trivial part of the proof is the verification of (L2) for the convergence in $D(X,Y)$.

Let $\xi_n \to \xi$ in $D(X,Y)$ and let $\{k_n\}$ be an increasing sequence of natural numbers. Let $x_n \to x$ in X. We shall put

$$p_m = \mathrm{Max}\{p \mid k_p \le m\} \quad \text{for } m \ge k_0$$

and prove that the sequence $\{x'_m\}$ in X defined

- up to $k_0 - 1$ arbitrarily,
- $x'_m = x_{p_m}$ for $m \ge k_0$

converges to x in X .

We shall apply (L3). If $\{x'_\ell\}$ is an arbitrary subsequence of $\{x'_n\}$ then the non-decreasing sequence $\{p_{\ell_m}\}$ of natural numbers has no constant subsequence. Hence it has an increasing subsequence $\{p_{\ell_{q_m}}\}$. Nevertheless, $\{x'_{p_{\ell_{q_m}}}\}$ is already a subsequence of both $\{x'_m\}$ and $\{x'_n\}$, therefore it converges to x in X . From $x_m = x'_{k_m}$ it follows

$\xi_{k_n}(x_n) = \xi_{k_n}(x'_{k_n}) \longrightarrow \xi(x)$ in $\mathcal{Y}$.

14. Let us denote by $\mathcal{X} \times \mathcal{Y}$ the product of the objects $\mathcal{X}$, $\mathcal{Y}$ in $\mathcal{L}$. By 6 it is the cartesian product of the underlying sets with the convergence: $(x_n, y_n) \longrightarrow (x, y)$ in $\mathcal{X} \times \mathcal{Y}$ iff $x_n \longrightarrow x$ in $\mathcal{X}$ and $y_n \longrightarrow y$ in $\mathcal{Y}$.

It is possible to prove by a simple calculation that in any category with products the functor $- \times - : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, assigning to a couple of morphisms $A_1 \xrightarrow{f} A_2$, $B_1 \xrightarrow{g} B_2 \in \mathcal{A}$ the morphism $f \times g : A_1 \times B_1 \longrightarrow A_2 \times B_2$ defined through the projections to the individual coordinates:

$$p_{A_2}^{A_2 \times B_2} \circ (f \times g) = f \circ p_{A_1}^{A_1 \times B_1} ,$$

$$p_{B_2}^{A_2 \times B_2} \circ (f \times g) = g \circ p_{B_1}^{A_1 \times B_1} ,$$

is associative, commutative, and has the singleton of $\mathcal{A}$ as a unity.

15. Proposition. $(\times, \mathbb{D})$ is a tensor couple in $(\mathcal{L}, \mathfrak{U}_{\mathcal{L}})$.

Proof. By the above remarks it suffices to concentrate on the adjunction which, defined by the same formula as in 12, clearly is correct.

Now let us assume that $(\otimes, \mathcal{H})$ is some tensor couple in $(\mathcal{L}, \mathfrak{U}_{\mathcal{L}})$. By 3.1 $\otimes$ must have the singleton $\mathbf{P}$ as a unity.

For any $\mathcal{X}, \mathcal{Y} \in |\mathcal{L}|$ we shall define mappings

$$s^{XY}: X \oplus Y \to X \otimes Y, \quad t^{XY}: X \otimes Y \to X \times Y ,$$

$$s^{XY}(x, y) = (c_x^{PX} \otimes c_y^{PY})(1)$$

and

$$t^{XY}(x, y) = (\kappa^X \circ (1_X \otimes {}_0^{YP})(x, y) ,$$

$$\ell^Y \cdot (c^{XP} \otimes 1_Y)(x, y)) , \quad x \in X , \ y \in Y$$

where 1 is the only element of $P \otimes P$.

The naturality of both s^{XY} and t^{XY} as well as the continuity of t^{XY} are obvious from the definitions. Observe that

$$\kappa^X \circ s^{XP}(x, 0) = \kappa_0^X(x, 0) = x, \quad \ell^X \circ s^{PX}(0, x) = \ell_0^X(0, x) = x ,$$

and $t^{XY} s^{XY}(x, y) = (x, y)$ for any $X, Y \in |\mathcal{L}|$, $x \in X$, $y \in Y$. If we show that s^{XY} is also always continuous, we shall have natural transformations $s: \oplus \longrightarrow \otimes$, $t: \otimes \longrightarrow \times$.

Let $(x_n, y_n) \longrightarrow (x, y)$ in $X \oplus Y$. We have

$$s^{XY}(x_n, y) = s^{XY} \circ (1_X \oplus c_y^{PX}) \circ \bar{\kappa}_0^X(x_n) =$$

$$= (1_X \otimes c_y^{PY}) \circ s^{XP} \circ \bar{\kappa}_0^X(x_n) = (1_X \otimes c_y^{PY}) \circ \bar{\kappa}^X(x_n) ,$$

$$s^{XY}(x, y_n) = s^{XY} \circ (c_x^{PX} \oplus 1_Y) \circ \bar{\ell}_0^Y(y_n) =$$

$$= (c_x^{PX} \otimes 1_Y) \circ s^{PY} \circ \bar{\ell}_0^Y(y_n) = (c_x^{PX} \otimes 1_Y) \circ \bar{\ell}^Y \quad .$$

From the continuity of $(1_X \otimes c_y^{PY}) \circ \bar{\kappa}^X$ and $(c_x \otimes 1_Y) \circ \bar{\ell}^Y$ it follows

(2) $s^{XY}(x_n, y) \to s^{XY}(x,y)$, $s^{XY}(x, y_n) \to s^{XY}(x,y)$

in $\mathcal{X} \otimes \mathcal{Y}$.

For any subsequence $\{s^{XY}(x_{k_n}, y_{k_n})\}$ of $\{s^{XY}(x_n, y_n)\}$ there is a subsequence $\{(x_{k_\ell}, y_{k_\ell})\}$ of $\{(x_{k_n}, y_{k_n})\}$ constant in one of the coordinates. The image of the latter under s^{XY} converges to $s^{XY}(x,y)$ in $\mathcal{X} \otimes \mathcal{Y}$ by (2). By (L3) we have $s^{XY}(x_n, y_n) \to s^{XY}(x,y)$ in $\mathcal{X} \otimes \mathcal{Y}$. s^{XY} is continuous.

16. Lemma. Identity is the only natural endotransformation of $\langle -,- \rangle_{\mathcal{L}}$.

Proof. For any $\sigma \in \langle \langle -,- \rangle_{\mathcal{L}}, \langle -,- \rangle_{\mathcal{L}} \rangle_{n.t.}$ and $X \xrightarrow{f} Y \in \mathcal{L}$ we have

(3) $$f \circ \sigma^{XX}(1_X) = \sigma^{XY}(f) = \sigma^{YY}(1_Y) \circ f \ .$$

In fact, (3) establishes a one-to-one correspondence between the endotransformations of $\langle -,- \rangle_{\mathcal{L}}$ and those of $1_{\mathcal{L}}$. But $1_{\mathcal{L}}$ has no other endotransformation than its identity since for any $v \in \langle 1_{\mathcal{L}}, 1_{\mathcal{L}} \rangle_{n.t.}$ and $X \in |\mathcal{L}|$ we have $v^X(x) = v^X . c_x^{PX}(0) = c_x^{PX} . v^P(0) = x$.

Consider the transformations $i = H \to B$, $j : D \to H$ conjugated to s and t respectively (see 3.4). By 16 they must be carried by identity, hence $H \leq B$ and $D \leq H$.

Let $X, Y, Z \in |\mathcal{L}|$. The commutative diagram

$$\begin{array}{ccc}
\langle X \otimes Y, Z \rangle & \xrightarrow{\ p^{XYZ}\ } & \langle X, H(Y,Z) \rangle \\
\langle s^{XY}, 1_Z \rangle \downarrow & & \downarrow \langle 1_X, i^{YZ} \rangle \\
\langle X \oplus Y, Z \rangle & \xrightarrow{\ \pi^{XYZ}\ } & \langle X, B(Y,Z) \rangle
\end{array}$$

shows that

a) s^{XY} is an epimorphism in $\mathcal{L}$,

b) we have $p^{XYZ} \varphi(x)(y) = \varphi s^{XY}(x, y)$ for any $X \otimes Y \xrightarrow{\varphi} Z$ $x \in X$, $y \in Y$.

17. Lemma. If s^{AA} is an isomorphism in $\mathcal{L}$, then $\oplus \approx \otimes$ and $H = B$.

Proof. It will do to prove $B \leq H$. $\oplus \approx \otimes$ will follow from the fact that the correspondence between mutually conjugated transformations is functorial.

1) First we show that $B(A,Y) \leq H(A,Y)$ for any $Y \in |\mathcal{L}|$. Assume that $\xi_n \to \xi$ in $B(A,Y)$. Then the mapping $\psi : A \longrightarrow B(A,Y)$ defined $\psi(n) = \xi_n$, $\psi(\omega) = \xi$ is continuous. Since $\langle s^{AA}, 1_Y \rangle$ is bijective, so is $\langle 1_A, i^{AY} \rangle$, and there is a mapping $A \xrightarrow{\psi'} H(A,Y)$ such that $\psi(a) = i^{AY} \psi'(a)$ for any $a \in A$. Hence

$\xi_n = \psi'(n) \longrightarrow \psi'(\omega) = \xi$ in $H(A,Y)$.

2) Let X be another object of $\mathcal{L}$. We know

- by 9 that $(A \xrightarrow{f} X)_{f \in |\mathcal{K}_{(A)}|}$ is a colimit in $\mathcal{L}$,

- that $H(-,Y)$ transfers colimits to limits,

- by 6 how the limits in $\mathcal{L}$ look like.

If $\xi_n \to \xi$ in $B(X,Y)$ then by the continuity of $B(f, 1_Y)$ for $f \in |\mathcal{K}_{(A)}(x)|$ we have always $\langle f, 1_Y \rangle \xi_n \to \langle f, 1_Y \rangle \xi$ in $B(A,Y) \leq H(A,Y)$.
The family $(H(X,Y) \xrightarrow{\langle f, 1_Y \rangle} H(A,Y))$ is a limit in $\mathcal{L}$. Thus $\xi_n \to \xi$ in $H(X,Y)$.

18. Lemma. If s^{AA} is not an isomorphism, then $\otimes \approx \times$ and $D = H$.

Proof. 1) Since s^{AA} is an epimorphism, for any $e \in A \otimes A$ there is a sequence $\{(a_n, b_n)\}$ in $A \oplus A$ such that $s^{AA}(a_n, b_n) \to e$ in $A \otimes A$. Put $(a, b) = t^{AA}(e)$.
If $(a_n, b_n) \to (a, b)$ in $A \oplus A$, then $s^{AA}(a_n, b_n) \to s^{AA}(a, b) = e$ in $A \otimes A$. Suppose $(a_n, b_n) \not\to (a, b)$ in $A \oplus A$. Then $a = b = \omega$ and there is an increasing sequence $\{k_n\}$ in ω such that $a_{k_n}, b_{k_n} \in \omega$ for $n \in \omega$. $\{a_{k_n}\}$ has an increasing subsequence $\{a_{k_{l_n}}\}$, $\{b_{k_{l_n}}\}$ has an increasing subsequence $\{b_{k_{l_{p_n}}}\}$. The mappings $A \xrightarrow{\varphi, \psi} A$ defined $\varphi(a_{k_{l_{p_{2n}}}}) = a_{k_{l_{p_{2m}}}}$, $\varphi(m) = \omega$ for $m \in \omega \setminus \{a_{k_{l_{p_{2n}}}} \mid n \in \omega\}$, $\varphi(\omega) = \omega$; $\psi(b_{k_{l_{p_{2n}}}}) = b_{k_{l_{p_{2n}}}}$, $\psi(m) = \omega$ for $m \in \omega \{b_{k_{l_{p_{2n}}}} \mid n \in \omega\}$, $\psi(\omega) = \omega$ are continuous. The convergent sequence $\{(\varphi \otimes \psi) s^{AA}(a_{k_{l_{p_n}}}, b_{k_{l_{p_n}}})\}$

has a common subsequence with both $\{s^{AA}(a_n, b_n)\}$ and the constant $\{s^{AA}(\omega, \omega)\}$. By (L0) and (L2) we have $s^{AA}(\omega, \omega) = e$. The mapping s^{AA} is onto.

2) If s^{AA} is not an isomorphism, the argument above shows that there are increasing sequences $\{c_n\}$, $\{d_n\}$ of natural numbers such that $s^{AA}(c_n, d_n) \to s^{AA}(\omega, \omega)$ in $A \otimes A$. We shall prove that for any $X, Y \in |\mathcal{L}|$ the mapping s^{XY} carries a morphism $X \times Y \to X \otimes Y$. Assume $x_n \to x$ in X, $y_n \to y$ in Y. By the same trick as in the proof of 13 we provide sequences $\{u\}$, $\{v\}$ such that $u_{c_n} = x_n$, $v_{d_n} = y_n$, $u_n \to x$ in X, $v_n \to y$ in Y. The mappings $A \xrightarrow{\vartheta} X$, $A \xrightarrow{\chi} Y$ defined $\vartheta(n) = u_n$, $\vartheta(\omega) = x$, $\chi(n) = v$, $\chi(\omega) = y$ are continuous. Hence $s^{XY}(x_n, y_n) =$

$$= (\vartheta \otimes \chi) s^{AA}(a_n, d_n) \to (\vartheta \otimes \chi) s^{AA}(\omega, \omega) = s^{XY}(x, y) \text{ in } X \otimes Y .$$

The transformation $t : \otimes \to \times$ has therefore a right inverse. From 3.4 it follows $H \leq D$, $\otimes \approx \times$.

19. <u>Theorem</u>. Up to a natural equivalence, there are exactly two tensor couples in $(\mathcal{L}, \mathcal{U}_{\mathcal{L}})$.

<u>Proof</u>. Observe that $A \oplus A < A \times A$. The rest is a consequence of the preceding lemmas.

20. <u>Remark</u>. Let $\mathcal{L}'$ be the category of all spaces for whose convergence we demand only the axiom (L1).

Let A' be the set $\omega + 1$ with the convergence: $a_n \to a$ in A' iff $a_n = a$ for all $n \in \omega$ or $a_n = n$, $a = \omega$.

For $X, Y \in |\mathcal{L}'|$ denote by $X \oplus' Y$ the cartesian product of the u derlying sets with the convergence: $(x_n, y_n) \to (x, y)$ in $X \oplus' Y$ if $x_n \to x$ in X, $y_n \to y$ in Y, and at least one of the sequences $\{x_n\}$, $\{y_n\}$ is a constant equal to its limit.

Substitute $\mathcal{L}'$ for $\mathcal{L}$, A' for A, and $\oplus'$ for $\oplus$ throughout the text and change the proposition 7 to " φ is an epimorphism in $\mathcal{L}'$ iff f is onto". Surveying the proofs (which, of course, contain quite a few places in this case superfluous), we can make sure that the results of this paper remain valid.

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