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A GENERALIZATION OF REFLEXIVE BANACH SPACES AND WEAKLY COMPACT OPERATORS

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Abstract

A Banach space X is almost reflexive if every bounded sequence in X contains a weak Cauchy subsequence. A continuous linear operator T: $X \longrightarrow Y$ is a weak Cauchy operator if it maps bounded sequences of X into sequences in Y which have a weak Cauchy subsequence. A comparison of this operator with other related operators is given along with certain properties of a Banach space involving the weak Cauchy operator. Conditions are given when the weak Cauchy operator is equivalent to other related operators.

1. Preliminaries. A Banach space X is said to be almost reflexive if every bounded sequence in X contains a weak Cauchy subsequence. A weakly complete space which is almost reflexive is reflexive. A reflexive space is always almost reflexive.

Let X and Y be Banach spaces and $T: X \longrightarrow Y$ a continuous linear operator. T is said to be a weak Cauchy operator if it maps bounded sequences of X into sequences

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in Y which have a weak Cauchy subsequence.

If Y is also weakly complete, then T is weakly compact. All weakly compact operators are weak Cauchy. T is said to be a completely continuous operator if it maps weak Cauchy sequences in X into norm convergent sequences in Y. X is said to be an unconditionally converging (uc operator) if it sends every weakly unconditionally converging (wuc) series in X into an unconditionally converging (uc) series in Y. X is said to be an ℓ_1-cosingular operator provided that for no Banach space E isomorphic to ℓ_1 does there exist epimorphisms $h_1 : X \longrightarrow E$ and $h_2 : Y \longrightarrow E$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "h_1"'] & & Y \arrow[dl, "h_2"] \\ & E & \end{tikzcd}$$

is commutative. T is ℓ_1-cosingular if and only if T', the conjugate of T, is a uc operator (see [3]).

2. Weak Cauchy, ℓ_1-cosingular, and uc operators

We now compare the operators weak Cauchy, ℓ_1-cosingular, and uc.

Proposition 2.1. If $T : X \longrightarrow Y$ is weak Cauchy, then T is ℓ_1-cosingular.

Proof: Assume that T is not an ℓ_1-cosingular operator, i.e. that there exist epimorphisms $h_1 : X \longrightarrow \ell_1$ and $h_2 : Y \longrightarrow \ell_1$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "h_1"'] & & Y \arrow[dl, "h_2"] \\ & \ell_1 & \end{tikzcd}$$

is commutative. Since T maps bounded sets into sets where every sequence has a weak Cauchy subsequence, then $h_1 = h_2 T : X \longrightarrow \ell_1$ must do the same. Let K denote the unit sphere of X. Since ℓ_1 is weakly complete, every sequence of $h_1(K)$ contains a weakly convergent subsequence. Hence h_1 is weakly compact, and since h_1 is an epimorphism, ℓ_1 must be reflexive. This contradiction completes the proof.

Corollary 2.2. If T is weak Cauchy, then T' is a uc operator.

Proposition 2.3. If $T' : Y' \longrightarrow X'$ is weak Cauchy, then T is a uc operator.

Proof: Assume T is not uc. By Lemma 1 of [2], the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] & & Y \\ & c_0 \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & \end{tikzcd}$$

is commutative where i_1 and i_2 are isomorphic embeddings.

Hence the diagram

$$\begin{tikzcd} Y' \arrow[rr, "T'"] \arrow[dr, "i_2'"'] & & X' \arrow[dl, "i_1'"] \\ & \ell_1 & \end{tikzcd}$$

is commutative where i_1' and i_2' are epimorphisms. Since weak Cauchy convergence implies norm convergence in ℓ_1, i_1' is completely continuous. Since T' is weak Cauchy, $i_2' = i_1'T'$ is compact. Now i_2' is onto, so ℓ_1 is finite dimensional. This contradiction shows that T must be uc.

Remark: From [3] we know that if $T' : Y' \longrightarrow X'$ is an ℓ_1-cosingular operator, then $T : X \longrightarrow Y$ is a uc operator. The following example shows that the converse is not true. This example was communicated to me by A. Pelczynski.

Example 2.4. If $T : X \longrightarrow Y$ is a uc operator, then T' is not necessarily an ℓ_1-cosingular operator.

Proof: Let X be a Banach space with a boundedly complete basis. Then by Theorem 1 of [4] there exists a separable space E such that $E'' = JE + F$ where JE is the natural image of E into E'' and where F is isomorphic to X.

Now put $X = \ell_1$ and $Y = E'$. Since E'' is separable, $Y = E'$ is separable. Hence Y does not contain a subspace isomorphic to c_0 because if a conjugate Banach space contains a subspace isomorphic to c_0, it contains a subspace isomorphic to m by Theorem 4 of [1] and hence Y could not be separable. Thus the identity operator $I : Y \longrightarrow Y$ is a uc operator but its conjugate I' is clearly not an ℓ_1-cosingular operator.

Remark: The identity operator $I : c_0 \longrightarrow c_0$ is weak Cauchy and ℓ_1-cosingular but not uc. $I' : \ell_1 \longrightarrow \ell_1$ is uc but not weak Cauchy and not ℓ_1-cosingular. $I'' : m \longrightarrow m$ is not weak Cauchy and not uc but is ℓ_1 -cosingular. Hence the converses of Propositions 2.1 and 2.3 are not true. Also if T is weak Cauchy, then T' is not necessarily weak Cauchy.

3. Weak Cauchy V and weak Cauchy V' properties.

We now consider spaces X which are such that the converses to Propositions 2.1 and 2.3 hold.

Definition 3.1. Let X be a Banach space. X has the weak Cauchy V property if it satisfies one of the following equivalent conditions:

(a) For every B -space Y, every uc operator $T: X \to Y$ is such that $T': Y' \to X'$ is weak Cauchy.

(b) Every subset K' of X' satisfying the condition

(+) $\lim_n \sup_{x' \in K'} x' x_n = 0$ for every wuc series $\sum_n x_n$ in X

has a weak Cauchy sequence.

Remark: The proof that (a) and (b) are equivalent is similar to the proof for Proposition 1 of [6]. X is said to have property V if for every B -space Y, every uc operator $T: X \to Y$ is weakly compact. X has weak Cauchy V property and X' is weakly complete if and only if X has property V (see Corollary 5 of [6]).

Proposition 3.2. Let X be weakly complete. Then X has weak Cauchy V if and only if X' is almost reflexive.

Proof: Since X is weakly complete, by Orlicz's theorem every wuc series in X is uc. Thus every bounded set in X' satisfies the condition (+). Since X has weak Cauchy V, every bounded set in X' has a weak Cauchy sequence. So X' is almost reflexive. The converse is clear.

Definition 3.3. Let Y be a Banach space. Y has the the weak Cauchy V' property if it satisfies one of the following equivalent conditions:

(c) For every B-space X, every ℓ_1-cosingular operator $T: X \longrightarrow Y$ is weak Cauchy.

(d) Every subset K of Y satisfying the condition

(+ +) $\lim_n \sup_{y \in K} y'_n y = 0$ for every wuc series $\sum_n y'_n$ in Y'

has a weak Cauchy sequence.

Remark: The proof that (c) and (d) are equivalent is similar to the proof for (a) and (b) in Definition 3.1 using the fact that T' is uc. Y is said to have property V' if for every B-space X, every ℓ_1-cosingular operator $T: X \longrightarrow Y$ is weakly compact. Y has weak Cauchy V' and Y is weakly complete if and only if Y has property V' (see Proposition 6 of [6]).

Proposition 3.4. Let Y' be weakly complete. Then Y has weak Cauchy V' if and only if Y is almost reflexive.

Proof: The proof is similar to the proof of Proposition 3.2.

Remark: By following [6], we have the following:
(A) Let X have weak Cauchy V' property. Then X is almost reflexive if and only if no complemented subspace of X is isomorphic to ℓ_1. (B) Let X have weak Cauchy V property. Then X' is almost reflexive if and only if no subspace of X is isomorphic to c_0.

Proposition 3.5. If X has weak Cauchy V then X'

has weak Cauchy V'; if X' has weak Cauchy V then X has weak Cauchy V'.

Proof: The proof follows from Definitions 3.1 and 3.3.

Remark: We show that the converses of Proposition 3.5 are not true. For properties V and V' this is not known (see [6]).

Example 3.6. If X' has weak Cauchy V', then X does not necessarily have weak Cauchy V.

Proof: Consider the space $X = E'$ given in Example 2.4. Since $I : E' \longrightarrow E'$ is uc but $I' : E'' \longrightarrow E''$ is not weak Cauchy, $X = E'$ does not have weak Cauchy V. But $X' = E'' = JE + F$ where F is isomorphic to ℓ_1 and both E and ℓ_1 have weak Cauchy V' property. Therefore X' has weak Cauchy V'.

Example 3.7. If X has weak Cauchy V', then X' does not necessarily have weak Cauchy V.

Proof: Consider the space $X = E'$ as given in Example 3.6. Since E'' is separable, E' is almost reflexive; therefore $X = E'$ has weak Cauchy V' property. Since $I : E'' \longrightarrow E''$ is uc but I' is not weak Cauchy, $X' = E''$ does not have the weak Cauchy V property.

Remark: The B-space E is an example which has weak Cauchy V but not property V. Also E has weak Cauchy V' but not property V'.

4. Dunford - Pettis property

A Banach space X is said to have the Dunford-Pettis (D.P.) property provided that for every Banach space Y, every weakly compact linear operator $T: X \longrightarrow Y$ is completely continuous.

Theorem 4.1. Let X be a Banach space. X has the D.P. property if and only if for every B-space Y, every weak Cauchy operator $T': Y' \longrightarrow X'$ is such that T is completely continuous.

Proof: ($\Longleftarrow$) This follows since if T is weakly compact, T' is weakly compact and hence T' is weak Cauchy.

($\Longrightarrow$) It suffices to show for every B-space Y, if T' is weak Cauchy, then $\lim_n \|Tx_n\| = 0$ for every weakly convergent to 0 sequence $\{x_n\}$. Let $\overline{\lim_n} \|Tx_n\| = \delta \geq 0$. Let y'_n with $\|y'_n\| = 1$ be such that $y'_n(Tx_n) = \|Tx_n\|$ for all n. Put $x'_n = T'y'_n$. Thus w.l.o.g. we assume $\{x'_n\}$ is a weak Cauchy sequence. We have

$$\overline{\lim_n}\, x'_n x_n = \overline{\lim_n}\,(T'y'_n)x_n = \overline{\lim_n}\, y'_n(Tx_n) = \overline{\lim_n}\,\|Tx_n\| = \delta .$$

We now show $\delta = 0$ where $\lim |x'_n x_n| = \delta$. Let $\{m\}$ be a subsequence of $\{n\}$ such that $|x'_m x_m| \leq \delta/2$. Since $\{x_n\}$ weakly converges to 0 such a subsequence $\{m\}$ exists. We have

$$x'_m x_m = (x'_m - x'_n)x_m + x'_n x_m .$$

Since $\{x'_m - x'_n\}$ weakly converges to 0, we obtain

$$\delta = \lim_m |x'_m x_m| \leq \overline{\lim_m}\,|(x'_m - x'_n)x_m| + \overline{\lim}\,|x'_n x_m| \leq \delta/2 .$$

Thus $\delta = 0$.

Corollary 4.2. Suppose X or X' has D.P. property and X' is almost reflexive. Then a sequence in X is weak Cauchy if and only if it is norm Cauchy.

Proof: If X' has D.P. property then so does X (see [7]); so it suffices to take X with the D.P. property. Since X' is almost reflexive, $I': X' \longrightarrow X'$ is weak Cauchy. By Theorem 4.1, $I: X \longrightarrow X$ is completely continuous and the result follows.

Corollary 4.3. Let X have weak Cauchy Y and D.P. properties, and let $T: X \longrightarrow Y$. Then the following are equivalent.

(a) T is uc,

(b) T' is weak Cauchy,

(c) T is completely continuous,

(d) T' is ℓ_1-cosingular.

Proof: (a)$\Longrightarrow$(b)$\Longrightarrow$(c) is clear. (c)$\Longrightarrow$(a) follows from Proposition 1.9 of [2]. To complete the proof it suffices to show (b)$\Longrightarrow$(d)$\Longrightarrow$(a). Now (b)$\Longrightarrow$(d) follows from Proposition 2.1 and (d)$\Longrightarrow$(a) is found in [3].

We now consider somewhat a dual notion for the D.P. property.

Theorem 4.4. Let Y be a Banach space. Y has the D.P. property if and only if for every $\mathcal{B}$-space X, every weak Cauchy operator $T: X \longrightarrow Y$ is such that T' is completely continuous.

Proof: ($\Longleftarrow$) By [7] it suffices to show for every weakly convergent to 0 sequence $\{y_n\}$ in Y and for every weakly convergent to 0 sequence $\{y'_n\}$ in Y', $\lim_n y'_n y_n = 0$. Let $\{y_n\}$ be an arbitrary weakly convergent to 0 sequence in Y. Consider the linear operator $T: c_0 \longrightarrow Y$ with $Te_n = y_n$ where e_n denotes the n-th unit vector in

c_0. Then $T': Y' \longrightarrow \ell_1$ is completely continuous. By the properties of T', $T'y'(e_n) = y'(Te_n) = y'(y_n)$ for every y' in Y'. Now let $\{y'_m\}$ be an arbitrary sequence in Y' weakly convergent to 0. Then $0 = \lim_m \|T'y'_m\| = \lim_m \sup_n |y'_m(y_n)|$. Hence $\lim_n y'_n y_n = 0$ and so Y has the D.P. property.

($\Longrightarrow$) Suppose $T: X \longrightarrow Y$ is weak Cauchy and Y has D.P. property. It suffices to show $\lim_n \|T'y'_n\| = 0$ for every weakly convergent to 0 sequence $\{y'_n\}$. Let x_n with $\|x_n\| = 1$ be such that $T'y'_n(x_n) = \|T'y'_n\|$ for all n. Put $y_n = Tx_n$. The rest of the proof is analogous to that given in the proof of Theorem 4.1.

<u>Corollary 4.5</u>. If X is almost reflexive and X or X' has D.P. property, then a sequence in X' is weak Cauchy if and only if it is norm Cauchy.

<u>Remark</u>: The proof of Corollary 4.5 is similar to that of Corollary 4.2. Using Corollaries 4.2 and 4.5 we have that if X' has D.P. and is almost reflexive, then weak Cauchy sequences correspond to norm Cauchy sequences in both X and X''.

<u>Corollary 4.6</u>. Let Y have weak Cauchy Y' and D.P. properties, and let $T: X \longrightarrow Y$. Then the following are equivalent.

(a) T' is uc,

(b) T is ℓ_1-cosingular.

(c) T is weak Cauchy.

(d) T' is completely continuous.

Proof: (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) is clear. (d) $\Longrightarrow$ (a) follows from Proposition 1.9 of [2].

Corollary 4.7. Let X have weak Cauchy Y, X' have D.P. property, $T: X \longrightarrow Y$ and $T': Y' \longrightarrow X'$. The following are equivalent.

(a) T is uc.

(b) T' is weak Cauchy.

(c) T is completely continuous.

(d) T' is ℓ_1-cosingular.

(e) T'' is uc.

(f) T'' is completely continuous.

Proof: The proof follows easily using Corollaries 4.3 and 4.6, Proposition 3.5, and the fact that X' has D.P. implies X has D.P.

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