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ON SEQUENCES OF CONTRACTIVE-LIKE MAPPINGS AND FIXED POINTS

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Let (E, d) be a complete metric space, and k a positive integer. Recently, Kečkić ([2]) investigates sequences of mappings $f_n : E^k \to E$ satisfying the contractive-like condition:

(1) $$d(f_n(u_1, u_2, \dots, u_k), f_{n+1}(u_2, u_3, \dots, u_{k+1}))$$

$$\le \sum_{i=1}^{k} q_i d(u_i, u_{i+1}) \text{ for all } u_1, u_2, \dots, u_{k+1} \in E ,$$

where $q_1, q_2, \dots, q_k$ are non-negative constants with $\sum_{i=1}^{k} q_i < 1$. On the other hand, in unifying both Banach's contraction principle and Kannan's fixed point theorem, Reich ([4]) considers a single self-mapping f of E satisfying the condition:

(2) $$d(f(x), f(y)) \le a d(x, f(x)) + b d(y, f(y)) + c d(x, y)$$
for all $x, y \in E$,

where a, b, c are non-negative constants with $a + b + c < 1$. The purpose of this note is to consider a combined

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condition of (1) and (2), and obtain a result which contains both Kečkić's and Reich's results as particular cases. In fact, we prove

Theorem. Let (E, d) be a complete metric space, k a positive integer, f_n a mapping from E^k to E for $n = 1, 2, 3, \dots$. Suppose that

$$(3)\quad d(f_n(u_1, u_2, \dots, u_k), f_{n+1}(u_2, u_3, \dots, u_{k+1}))$$
$$\le \sum_{i=1}^{k} \{\alpha_i d(u_i, f_n(u_1, u_2, \dots, u_k)) + \beta_i d(u'_{i+1}, f_{n+1}(u_2, u_3, \dots, u_{k+1}))$$
$$+ \gamma_i d(u_i, u_{i+1})\} + \delta_n \quad \text{for } n = 1, 2, 3, \dots,$$

and for $u_1, u_2, \dots, u_{k+1} \in E$,

where $\alpha_i, \beta_i, \gamma_i, \delta_n$ are non-negative constants and
$$\sum_{i=1}^{k} [(k-i+1)(\alpha_i + \beta_i) + \gamma_i] < 1, \quad \sum_{n=1}^{+\infty} \delta_n < +\infty .$$

Define $x_{n+k} = f_n(x_n, x_{n+1}, \dots, x_{n+k-1})$ for $n = 1, 2, 3, \dots$,

where $x_1, x_2, \dots, x_k$ are chosen arbitrarily from E . Then

(i) $\{x_{n+k}\}$ converges in E ;

(ii) if $f_n(u, u, \dots, u) \to f(u, u, \dots, u)$ as $n \to +\infty$ for each $u \in E$, then $f(u, u, \dots, u) = u$ has x as a solution, where $x = \lim_{n \to +\infty} x_{n+k}$; if, furthermore,

$\alpha_i = \alpha_{k-i+1}$, $\beta_i = \beta_{k-i+1}$ for $i = 1, 2, 3, \dots, [\frac{k}{2}]$,

then x is unique to satisfy $f(u, u, \dots, u) = u$;

(iii) if f and α_i, β_i and x are as in (ii), and $f_n(y_n, y_n, \dots, y_n) = y_n$ for $n = 1, 2, 3, \dots$, then $y_n \to x$, the unique solution of $f(u, u, \dots, u) = u$.

Proof. To show that $\{x_{n+k}\}$ converges in E, it suffices to show that $\{x_{n+k}\}$ is a Cauchy sequence in E since E is complete. To this end, let $D_m = d(x_m, x_{m+1})$ for $m = 1, 2, 3, \dots$. Then $n = 1, 2, 3, \dots$, using (3) and the triangle inequality, one has

$$D_{n+k} \leq \sum_{i=1}^{k} \{\alpha_i d(x_{n+i-1}, x_{n+k}) + \beta_i d(x_{n+i}, x_{n+k+1}) + \gamma_i d(x_{n+i-1}, x_{n+i})\} + \delta_n$$

$$\leq \sum_{i=1}^{k} \{\alpha_i [D_{n+i-1} + D_{n+i} + \dots + D_{n+k-1}] + \beta_i [D_{n+i} + D_{n+i+1} + \dots + D_{n+k}] + \gamma_i D_{n+i-1}\} + \delta_n .$$

Hence by simple calculations one has

$$\sum_{n=1}^{m} D_{n+k} \leq s \sum_{i=1}^{k} D_i + s \sum_{n=1}^{m} D_{n+k} + \sum_{n=1}^{m} \delta_n ,$$

where $s = \sum_{i=1}^{k} [(k-i+1)(\alpha_i + \beta_i) + \gamma_i] < 1$. Therefore

$$\sum_{n=1}^{m} D_{n+k} \leq \frac{1}{1-s} [s \sum_{i=1}^{k} D_i + \sum_{n=1}^{m} \delta_n] \leq \frac{1}{1-s} [s \sum_{i=1}^{k} D_i + \sum_{n=1}^{+\infty} \delta_n] ,$$

of which the right hand side is independent of m. Thus, we conclude that $\sum_{n=1}^{+\infty} D_{n+k}$ is a convergent series. Now, for $m > n$, $d(x_{n+k}, x_{m+k}) \leq \sum_{i=n}^{m} D_{i+k} \to 0$ as

$n \to +\infty$, proving that $\{x_{n+k}\}$ is a Cauchy sequence.

To show that $f(x, x, \dots, x) = x$, where $x = \lim_{n\to+\infty} x_{n+k}$, let us denote

$x^{i}_{m+k} = f_{m+i}(x_{m+i}, x_{m+i+1}, \dots, x_{m+k-1}, x, x, \dots, x)$

for $m = 1, 2, 3, \dots$, $i = 0, 1, 2, \dots, k$. Note that, in particular, $x^{0}_{m+k} = f_m(x_m, x_{m+1}, \dots, x_{m+k-1}) = x_{m+k}$, $x^{k}_{m+k} = f_{m+k}(x, x, \dots, x)$. Then for $m = 1, 2, 3, \dots$, we have

$$\text{(A)}\quad d(x, f(x, x, \dots, x))$$
$$\leq d(x, x^{0}_{m+k}) + d(x^{0}_{m+k}, x^{1}_{m+k}) + \dots + d(x^{k-1}_{m+k}, x^{k}_{m+k})$$
$$+ d(f_{m+k}(x, x, \dots, x), f(x, x, \dots, x))$$

by the triangle inequality. Using (3) and then the triangle inequality, if we denote

$$\Delta^{j} = \Delta^{j}_{m} = d(x^{j}_{m+k}, x^{j+1}_{m+k}) \quad \text{for } j = 0, 1, 2, \dots, k-1 ,$$

then we obtain

$$\text{(B)}\quad \begin{aligned} \Delta^{0} &\leq \frac{1}{1-\beta}\Big[\sum_{\ell=m}^{m+k-1} D_\ell + d(x, x^{0}_{m+k}) + d(x, x_{m+k-1}) + \delta_m\Big] \\ \Delta^{j} &\leq \frac{1}{1-\beta}\Big[\sum_{\ell=m}^{m+k-1} D_\ell + d(x, x^{0}_{m+k}) + d(x, x_{m+k-1}) + \delta_{m+j} \\ &\qquad + (\Delta^{0} + \Delta^{1} + \dots + \Delta^{j-1})\Big] \end{aligned}$$

for $j = 1, 2, 3, \dots, k-1$,

where $\beta = \sum_{i=1}^{k} \beta_i < 1$.

Given $\varepsilon > 0$, choose m so large that

$$\sum_{\ell=m}^{m+k-1} D_\ell < \frac{\varepsilon_1}{3},\ d(f_{m+k}(x,x,\dots,x),\ f(x,x,\dots,x)) < \frac{\varepsilon_1}{3},$$

$d(x, x_{n+k-1}) < \frac{\varepsilon_1}{6}, \delta_n < \frac{\varepsilon_1}{3}$ for all $n \geq m$, where

$$\varepsilon_1 = \frac{(1-\beta)^k \varepsilon}{A} \text{ with } A = 2\sum_{j=0}^{k-1}\left(1 + \frac{j(j+1)}{2}\right),$$

noting that such an m can be chosen since

$$\sum_{\ell=1}^{+\infty} D_\ell < +\infty,\ f_n \to f,\ x_{n+k} \to x \text{ and } \sum_{n=1}^{+\infty} \delta_n < +\infty .$$

Then from (B) we have

$$\Delta^j < \frac{\varepsilon}{A}\left(1+\frac{j(j+1)}{2}\right)(1-\beta)^{k-(j+1)} \leq \frac{\varepsilon}{A}\left(1+\frac{j(j+1)}{2}\right)$$

for $j = 0, 1, 2, \dots, k-1$,

and hence we have from (A) that

$$d(x, f(x,x,\dots,x)) < \frac{\varepsilon_1}{6} + \frac{\varepsilon_1}{3} + \Delta^0 + \Delta^1 + \dots + \Delta^{k-1}$$

$$< \frac{\varepsilon_1}{2} + \frac{\varepsilon}{A}\sum_{j=0}^{k-1}\left(1+\frac{j(j+1)}{2}\right) = \frac{\varepsilon_1}{2} + \frac{\varepsilon}{2} < \varepsilon .$$

As ε is arbitrary, we conclude that $f(x,x,\dots,x) = x$.

To prove the uniqueness, suppose that $y = f(y,y,\dots,y)$. For convenience, let $\bar{x}_m^i = f_{m+i}(x,x,\dots,x,y_1,y_2,\dots,y_i)$ for $i = 0,1,2,\dots,k$, $m = 1,2,3,\dots$, where $y_1 = y_2 = \dots = y_i = y$, noting that $\bar{x}_m^0 = f_m(x,x,\dots,x)$, $\bar{x}_m^k = f_{m+k}(y,y,\dots,y)$. Also, let $\bar{\Delta}^i = \bar{\Delta}_m^i = d(\bar{x}_m^i, \bar{x}_m^{i+1})$ for $i = 0,1,2,\dots,k-1$. Then by the triangle inequality,

(C) $d(x,y) = d(f(x,x,\ldots,x), f(y,y,\ldots,y))$

$$\le d(f(x,x,\ldots,x), f_m(x,x,\ldots,x)) + d(f(y,y,\ldots,y),$$

$$f_{m+k}(y,y,\ldots,y)) + \bar{\Delta}^0 + \bar{\Delta}^1 + \bar{\Delta}^2 + \ldots + \bar{\Delta}^{k-1} .$$

Now, by (3) and then by the triangle inequality again, we have

$$\bar{\Delta}^0 \le \sum_{i=1}^{k} \alpha_i d(x, \bar{x}_m^0) + \sum_{i=1}^{k-1} \beta_i [d(x, \bar{x}_m^0) + \bar{\Delta}^0]$$

$$+ \beta_k [d(y, \bar{x}_m^k) + \bar{\Delta}^1 + \bar{\Delta}^2 + \ldots + \bar{\Delta}^{k-1}] + \gamma_k d(x,y) + \delta_m ;$$

$$+ \sum_{i=1}^{k-2} \beta_i [d(x, \bar{x}_m^0) + \bar{\Delta}^0 + \bar{\Delta}^1] + (\beta_k + \beta_{k-1})[d(y, \bar{x}_m^k) + \bar{\Delta}^2 + \bar{\Delta}^3 + \ldots + \bar{\Delta}^{k-1}]$$

$$+ \sum_{i=1}^{k-2} \beta_i [d(x, \bar{x}_m^0) + \bar{\Delta}^0 + \bar{\Delta}^1] + (\beta_k + \beta_{k-1})[d(y, \bar{x}_m^k) + \bar{\Delta}^2 + \bar{\Delta}^3 + \ldots + \bar{\Delta}^{k-1}]$$

$$+ \gamma_{k-1} d(x,y) + \delta_{m+1} ;$$

$$\bar{\Delta}^{k-1} \le \alpha_1 [d(x, \bar{x}_m^0) + \bar{\Delta}^0 + \bar{\Delta}^1 + \ldots + \bar{\Delta}^{k-2}]$$

$$+ \sum_{i=2}^{k} \alpha_i [d(y, \bar{x}_m^k) + \bar{\Delta}^{k-1}] + \sum_{i=1}^{k} \beta_i d(y, \bar{x}_m^k)$$

$$+ \gamma_1 d(x,y) + \delta_{m+k-1} .$$

Hence, using the condition $\alpha_i = \alpha_{k-i+1}$, $\beta_i = \beta_{k-i+1}$ **for** $i = 1, 2, \ldots, [\frac{k}{2}]$, **we have**

$$\bar{\Delta}^0 + \bar{\Delta}^1 + \ldots + \bar{\Delta}^{k-1} \le \frac{1}{1-(t-(\alpha+\beta))} \{ t [d(x, \bar{x}_m^0) + d(y, \bar{x}_m^k)] + \gamma d(x,y) + \sum_{i=1}^{k} \delta_{m+i-1} \} ,$$

where $\alpha = \sum_{i=1}^{k} \alpha_i$, $\beta = \sum_{i=1}^{k} \beta_i$, $\gamma = \sum_{i=1}^{k} \gamma_i$ **and** $t = \sum_{i=1}^{k} (k-i+1)(\alpha_i + \beta_i)$.

Hence from (C) we have

$$\left(1 - \frac{\gamma}{1-(t-(\alpha+\beta))}\right) d(x,y) \leq A(m) \quad ,$$

where $A(m)$ is an expression such that $A(m) \to 0$ as $m \to +\infty$.

Noting that $1 - \dfrac{\gamma}{1-(t-(\alpha+\beta))} > 0$ by the condition that $\sum_{i=1}^{k} [(k-i+1)(\alpha_i + \beta_i) + \gamma_i] < 1$, we conclude that $d(x,y) = 0$, so that $x = y$, proving the uniqueness in part (ii).

With a slight modification in the proof of uniqueness, one proves easily that $d(y_m, x) \to 0$ as $m \to +\infty$, so that $y_m \to x$ as $m \to \infty$ follows.

<u>Remarks</u>. (I) Reich's result in [4] is obtained from our theorem (part (ii)) by taking $k = 1$, $f = f_m$ for $m = 1, 2, 3, \ldots$.

(II) If $\alpha_i = \beta_i = 0$ for $i = 1, 2, 3, \ldots, k$, then one sees that $f_m(u, u, \ldots, u) \to f(u, u, \ldots, u)$ uniformly, so that our result (part (i) and (ii)) contains that of Kečkić's in [2].

(III) Part (iii) gives some kind of sufficient conditions for a sequence of fixed points of functions to converge to the fixed point of the convergent function. For other kinds of sufficient conditions, we refer to Bonsall [1], Nadler [3], and also Singh and Russell [5].

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R e f e r e n c e s

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