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# A NOTE ON FRÉCHET SPACES [1)]

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Recall that a Fréchet space $(L, \lambda)$ is a $T_1$ topological space such that for every subset $A$ we have $\lambda A = \{x \mid x = \lim x_n , x_n \in A\}$, i.e. $\lambda A$ is the set of all limit points of sequences of points of $A$; the space $(L, \lambda)$ is said to be sequentially regular if for every sequence $\langle x_n \rangle$ of points of $L$ and every point $x$ such that $x \in L - \lambda U(x_n)$ there is a continuous function f on $(L, \lambda)$, $0 \leq f(\mathcal{S}) \leq 1$, and a subsequence $\langle n_i \rangle$ of $\langle n \rangle$ such that $f(x) = 0$, $f[U(x_{n_i})] = 1$ (cf.[3]).

Following [5] a $T_1$ topological space $(L, \lambda)$ is called $\aleph_0$-regular if for every countable subset $A$ and every point $x$ such that $x \in L - \lambda A$ there is a continuous function $f$ on $(L, \lambda)$, $0 \leq f(x) \leq 1$ such that $f(x) = 0$, $f[A] = 1$. It can be readily seen that every $\aleph_0$-regular Fréchet space is sequentially regular. J. Novák asked in [5] whether every sequentially regular Fréchet space is $\aleph_0$-regular.

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1) The article is a part of [1].

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The main purpose of the present paper is to show that the answer is no. The space $\Lambda_\infty$ constructed by F.B. Jones in [2] [2] (as a Moore space which is not completely regular) is a counter-example. We also give a necessary and sufficient condition for a Fréchet sequentially regular space to be $\aleph_0$-regular and two sufficient conditions for an $\aleph_0$-regular Fréchet space to be completely regular.

Example. Let $L$ be the subset of all points $(x, y)$ of the Euclidean plane $R \times R$ such that $y \geqq 0$ provided with the following refinement of the product topology: for $r > 0$, the sets

$$V^r(x, 0) = \{(x, 0)\} \cup \{(u, v) \mid (u, v) \in L, (u-x)^2 + (v-r)^2 < r^2\}$$

are also neighbourhoods of the point $(x, 0)$ (Niemytzky space).

Denote by $\lambda$ the just described topology. Clearly, $(L, \lambda)$ satisfies the first axiom of countability and hence it is Fréchet. The subspace $(D, \lambda/D)$ of $(L, \lambda)$ where $D = \{(x, 0) \mid x \in R\}$, is discrete. The space $(L, \lambda)$ is completely regular and hence sequentially regular. The set $D$ is the union of two disjoint uncountable sets, denote them by $A$ and by $B$, such that if $U$ is an open set containing uncountably many points of one of them, then $\lambda U$ contains uncountably many points of the other (for the proof see [2]).

Let $\langle (L_m, \lambda_m) \rangle_{m=1}^{\infty}$ be a simple sequence of disjoint copies of the space $(L, \lambda)$. For convenience we may

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2) It is Professor J. Novák who called my attention to that article.

imagine these spaces as lying in different planes of the three-dimensional Euclidean space parallel to the plane of $L$. For each point set $H$ in $L$ and to every natural $m$ there corresponds in a natural way the set $H_m$ in $L_m$ (the set $H$ is the projection of every $H_m$). The symbol $q$ denotes always a point of $D$.

Let $\sum_{m=1}^{\infty} (L_m, \lambda_m)$ be the topological sum of the above sequence. We modify it in the following manner:

1. If $m$ is odd $(m = 1, 3, 5, \dots)$ and $q$ is a point of $B$, then we identify points $q_m$ and $q_{m+1}$ to $(q_m; q_{m+1})$; if $m$ is even $(m = 2, 4, 6, \dots)$ and $q$ is a point of $A$, then we identify points $q_m$ and $q_{m+1}$ to $(q_m; q_{m+1})$ (the projection of $(q_m; q_{m+1})$ is $q$ in this case). Let for $k > 0$ the sets

$$W^k((q_m; q_{m+1})) =$$

$$= \{(q_m; q_{m+1})\} \cup \{V_m^k(q) - (q)\} \cup \{V_{m+1}^k(q) - (q_{m+1})\}$$

be fundamental systems of neighbourhoods of these points, i.e. we take a quotient space of $\sum_{m=1}^{\infty} (L_m, \lambda_m)$.

2. We add one "ideal" point $p$ (distinct from all) to the modified $\sum_{m=1}^{\infty} (L_m, \lambda)$.

Let for $k = 1, 2, 3, \dots$, the sets

$$O_k(p) = (p) \cup \{\bigcup_{m>k} \bigcup_{y>0} (x_m, y_m)\} \cup \{\bigcup_{m>k} (q_m; q_{m+1})\}$$

form a fundamental system of neighbourhoods of $p$.

Denote by $(L_\infty, \lambda_\infty)$ this modified space (cf. [2], where $\Lambda_\infty = (L_\infty, \lambda_\infty)$ ). The space $(L_\infty, \lambda_\infty)$ satisfies the first axiom of countability and hence it is

Fréchet, it is "completely regular at every point" except $p$ but it is not completely regular (at $p$ ) since $p \in L_\infty - \lambda_\infty A_1$ , but for each continuous function f on $(L_\infty, \lambda_\infty)$ we have $f(p) \in \overline{f[A_1]}$ (cf.[2]).

<u>Proposition</u>. The Fréchet space $(L_\infty, \lambda_\infty)$ is sequentially regular but fails to be $\aleph_0$-regular.

<u>Proof</u>. First prove that $(L_\infty, \lambda_\infty)$ is sequentially regular. Since $(L_\infty, \lambda_\infty)$ is "completely regular and hence sequentially regular at every point" except $p$ , we have to prove that if $\langle x_m \rangle$ is a sequence of points of $L_\infty$ such that $p \in L_\infty - \lambda_\infty \bigcup_{m=1}^{\infty} (x_m)$ , then there is a continuous function f on $(L_\infty, \lambda_\infty)$ and a subsequence $\langle x_{m_i} \rangle$ of $\langle x_m \rangle$ such that

$$f(p) = 0, \quad f(x_{m_i}) = 1, \quad i = 1, 2, 3, \dots .$$

Since there is a natural $k_0$ such that $x_m \in L_\infty - O_{k_0}(p)$ for all $m$ , we always can and do select a subsequence $\langle x'_{m_i} \rangle$ of $\langle x_m \rangle$ such that

a) $\langle x'_{m_i} \rangle$ is a constant sequence or the projection of no $x'_{m_i}$ lies in $D \subset L$ . In this case the construction of f and the subsequence $\langle x_{m_i} \rangle$ of $\langle x'_{m_i} \rangle$ and hence of $\langle x_m \rangle$ is easy and is omitted.

b) If $(x'_i, 0) \in D \subset L$ is the projection of $x'_{m_i}$ , i.e. $x'_{m_i}$ is either of the form of $(q^{(i)}_{k m}; q^{(i)}_{k m+1})$, $m \leqq k_0$ , or $x'_{m_i} \in A_1$ , then there is a strictly monotone, say increasing, subsequence $\langle x_i \rangle$ of the sequence $\langle x'_i \rangle$ of real numbers $x'_i$ . Let $\langle \kappa_i \rangle$ be a sequence of positive real numbers such that

$x_{i-1}+r_{i-1} < x_i - r_i < x_i + r_i < x_{i+1} - r_{i+1}$, $i = 1, 2, 3, \dots$ .

Denote by $U(x_{m_i}) = (V^{r_i}(x_i, 0))_1$ if $x_{m_i} \in A_1$ and

$$U(x_{m_i}) = W^{r_i}((q_m^{(i)}; q_{m+1}^{(i)}))$$

otherwise. Now, let f be a function on $(L_\infty, \lambda_\infty)$ defined in the following manner:

$f(x) = 1$ for $x = x_{m_i}$ ;

$f(x) = 0$ for each $x$ on the boundary of the neighbourhood $U(x_{m_i})$ of $x_{m_i}$ and linear on the segment from $x_{m_i}$ to $x$, $i = 1, 2, 3, \dots$ ;

$f(x) = 0$ for $x \in L_\infty - \lambda \bigcup_{i=1}^{\infty} U(x_{m_i})$ .

It is easy to verify that f has the desired properties. If the sequence $\langle x_i \rangle$ is decreasing, then the procedure is similar.

Secondly, denote by

$$C = \{(x, y) \mid (x, y) \in L - D;\ x, y \text{ rational}\} .$$

The set $C_1$ is countable and can be arranged into a sequence $\langle x_m \rangle$ and $p \in L_\infty - \lambda_\infty \bigcup_{m=1}^{\infty} (x_m)$ . As $A_1 \subset \lambda_\infty \bigcup_{m=1}^{\infty} (x_m)$ , we have $f(p) \in \overline{\bigcup_{m=1}^{\infty} (f(x_m))}$ for each continuous function f on $(L_\infty, \lambda_\infty)$ . Therefore $(L_\infty, \lambda_\infty)$ fails to be $\aleph_0$-regular. This completes the proof.

Let $(L, \lambda)$ be a Fréchet sequentially regular space. Recall that the completely regular modification $\tilde{\lambda}$ of $\lambda$ is the finest of all completely regular topologies for L coarser than $\lambda$ , the systems of continuous functions on

$(L, \lambda)$ and on $(L, \tilde{\lambda})$ coincide and $\lim x_n = x$ if and only if the sequence $\langle x_n \rangle$ is eventually in every $\tilde{\lambda}$ -neighbourhood of $x$ (see [3]). A point $x_0$ is called a side-point of a sequence $\langle x_n \rangle$ in $(L, \tilde{\lambda})$ if any subsequence $\langle x_{n_i} \rangle$ of $\langle x_n \rangle$ does not converge to $x_0$ and the sequence $\langle x_n \rangle$ is frequently in every $\tilde{\lambda}$ -neighbourhood of $x_0$ .

<u>Theorem 1</u>. A Fréchet sequentially regular space $(L, \lambda)$ is $\mathcal{X}_0$ -regular if and only if there is no sequence in $(L, \tilde{\lambda})$ having a side-point, where $\tilde{\lambda}$ is the completely regular modification of $\lambda$ .

<u>Proof</u>. I. If there is a sequence $\langle x_n \rangle$ in $(L, \tilde{\lambda})$ having a side-point $x_0$ , then

$$x_0 \in L - \lambda \, U(x_n) \, , \quad x_0 \in \tilde{\lambda} \, U(x_n) \quad .$$

Thus for each continuous function $f$ on $(L, \tilde{\lambda})$ and hence, as mentioned above, on $(L, \lambda)$ we have

$$f(x_0) \in \overline{U(f(x_n))} \quad .$$

But this implies that $(L, \lambda)$ cannot be $\mathcal{X}_0$ -regular.

II. If $(L, \lambda)$ is not $\mathcal{X}_0$ -regular, then there is a sequence $\langle x_n \rangle$ of points $x_n \in L$ and a point $x_0 \in L$ such that

$$x_0 \in L - \lambda \, U(x_n)$$

and for each continuous function $f$ on $(L, \lambda)$ there is a subsequence $\langle n_i \rangle$ of $\langle n \rangle$ such that

$$\lim f(x_{n_i}) = f(x_0) \quad .$$

From the definition of $\tilde{\lambda}$ it follows that

$$x_0 \in \tilde{\lambda}\, U(x_n) \quad ,$$

i.e. $x_0$ is a side-point of the sequence $\langle x_n \rangle$ in $(L, \tilde{\lambda})$.

Theorem 2. A regular separable $\aleph_0$ -regular Fréchet space $(L, \lambda)$ is completely regular.

Proof. Denote by $S \subset L$ a countable set such that $G \cap S \neq \emptyset$ for each non-empty open set $G \subset L$. Let $F \subset L$ be a non-empty closed set and $x_0 \in L - F$. Then there is a neighbourhood $W(x_0)$ such that $\lambda W(x_0) \subset \subset L - F$ and $(L - \lambda W(x_0)) \cap S \neq \emptyset$. Hence $(L - W(x_0)) \cap S \neq \emptyset$. Now, arrange the countable set $(L - W(x_0)) \cap S$, either finite or infinite, into a sequence $\langle x_n \rangle$. Evidently

$$x_0 \in (L - \lambda U(x_n)) \subset L - F \quad .$$

Since $(L, \lambda)$ is $\aleph_0$ -regular, there is a continuous function $f$ on $(L, \lambda)$ such that

$$f(x_0) = 0 \, , \; f[U(x_n)] = 1 = f[F] \quad .$$

Corollary. A first-countable separable $\aleph_0$ -regular topological space is completely regular.

Proof. Professor J. Novák proved in [4] that every first-countable sequentially regular topological space is regular. The assertion follows at once from the foregoing Theorem 2.

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