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ON REAL SUBMANIFOLDS of $\mathbb{C}^2$ and $\mathbb{C}^3$

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(Preliminary communication)

In what follows, I determine the structure of some real submanifolds of $\mathbb{C}^2$ and $\mathbb{C}^3$ which are invariant under a transitive group of holomorphic mappings. The full exposition is to be published in Czech.Math.J. The results were obtained during my stay at the universities of Delhi, Chandighar and Bombay and at the Tata Inst. of Fundamental Research in Bombay under the Czechoslovak-Indian Cultural Exchange Programme.

1. In $\mathbb{C}^n$, consider the coordinates $(z_1,\dots,z_n)$, $z_i = x_i + i y_i$. Let $\iota : \mathbb{C}^n \to \mathbb{R}^{2n}$ be the usual identification $\iota(z_1,\dots,z_n) = (x_1, y_1, \dots, x_n, y_n)$. In $\mathbb{R}^{2n}$, we have the well known induced endomorphism $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $J^2 = -id.$, given by $J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$, $J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}$. Denote by Γ the pseudogroup of all local holomorphic diffeomorphisms in $\mathbb{C}^n$ (or $\iota(\Gamma)$ in $\mathbb{R}^{2n}$ resp.), let $\Gamma_b \subset \Gamma$ be the sub-pseudogroup of maps $z'_i = f_i(z_1,\dots,z_n)$ satisfying

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$$\left| \det \frac{\partial(z'_1, \ldots, z'_n)}{\partial(z_1, \ldots, z_n)} \right| = 1 .$$

Let $M^m \subset \mathbb{C}^n$ be a real submanifold, let us write again M^m instead of $\iota(M^m)$. Consider a point $p \in \in M^m$, the tangent space $T_p \equiv T_p(M^m)$, and define τ_p as $T_p \cap J T_p$. Let $v_0 \in \tau_p$. In a neighbourhood $\mathcal{O} \subset \subset M^m$ of p, consider a vector field v such that $v_p = v_0$ and $v_q \in \tau_q$ for each $q \in \mathcal{O}$. The map $L_p^{(1)} : : \tau_p \longrightarrow T_p / \tau_p$ be given by $L_p^{(1)}(v_0) = \pi_1([v, Jv]_p)$, where $\pi_1 : T_p \longrightarrow T_p / \tau_p$ is the projection; $L_p^{(1)}(v_0)$ depends on v_0 only. Let $\sigma_p \subset T_p$ be the linear hull of the set $\pi_1^{-1}(L_p^{(1)}(\tau_p))$. The map $L_p^{(2)} : \tau_p \longrightarrow T_p / \sigma_p$ be defined by $L_p^{(2)}(v_0) = \pi_2([v, [v, Jv]]_p)$, $\pi_2 : : T_p \longrightarrow T_p / \sigma_p$ being the projection; $L_p^{(2)}$ depends only on v_0 as well. $L_p^{(1)}$ and $L_p^{(2)}$ are the so-called Levi maps.

Write $G(M^m) = \{\gamma \in \Gamma \mid \gamma(M^m) = M^m\}$ and $G_s(M^m) = G(M^m) \cap \Gamma_s$.

2. Consider the case $n = 3$, $m = 4$ and the pseudogroup Γ. Suppose $\dim \tau_p = 2$, $L_p^{(1)} \not\equiv 0$, $L_p^{(2)} \not\equiv 0$ at each point $p \in M^4$. If $G(M^4)$ is transitive on M^4, $G(M^4)$ is a Lie group and $\dim G(M^4) \leq 5$. Let us consider a manifold M^4 with $\dim G(M^4) = 5$ and the manifold N^4 given by

(1) $\bar{z}_2 - z_2 = i(\bar{z}_1 - z_1)^2 \, , \; \bar{z}_3 - z_3 = (\bar{z}_1 - z_1)^3 \, .$

If $p \in M^4$, $q \in N^4$ are arbitrary points, there is a neighbourhood $\mathcal{O} \subset M^4$ of p and a $\gamma \in \Gamma$ such that $\gamma(\mathcal{O}) \subset N^4$, $\gamma(p) = q$, i.e., M^4 and N^4 are locally Γ-equivalent. The group $G(N^4)$ is

(2) $$z_1' = a z_1 + b + c i \quad ,$$

$$z_2' = 4ac z_1 + a^2 z_2 + d + 2c^2 i \quad ,$$

$$z_3' = -12ac^2 z_1 - 6a^2 c z_2 + a^3 z_3 + f - 4c^3 i$$

where $a, b, c, d, f \in \mathbb{R}$.

3. Consider the case $n = 2, m = 3$ and the pseudogroup Γ_h . Then $\dim \tau_p = 2$; suppose $L_p^{(1)} \neq 0$ at each point $p \in M^3$. If $G_h(M^3)$ is transitive on M^3 , then it is a Lie group with $\dim G_h(M^3) \leq 4$. Consider the manifolds N_r^3 , N_R^3 , N_0^3 given successively by

(3) $$z_1 \bar{z}_1 + z_2 \bar{z}_2 = r^2 \quad (r > 0) \quad ,$$

(4) $$z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2R \quad (R > 0) \quad ,$$

(5) $$i(z_2 - \bar{z}_2) = (z_1 - \bar{z}_1)^2 \quad .$$

Let $\dim G_h(M^3) = 4$. Then there is exactly one manifold among the manifolds N_r^3 , N_R^3 , N_0^3 - denote it by N^3 - with the following property:

Choose $p \in M^3, q \in N^3$, then there is a neighbourhood $\mathcal{O} \subset M^3$ of p and a $\gamma \in \Gamma_h$ such that $\gamma(p) = q$,

$\gamma(\sigma) \subset N^3$. The groups $G_p(N^3_k)$, $G_p(N^3_R)$ and $G_p(N^3_0)$ are given by

(6) $z_1' = \alpha z_1 - \beta z_2$,

$z_2' = e^{ia}(\bar{\beta} z_1 + \bar{\alpha} z_2)$,

where $\alpha, \beta \in \mathbb{C}$, $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$, $a \in \mathbb{R}$;

(7) $z_1' = e^{if}(az_1 + ibz_2)$,

$z_2' = e^{if}(icz_1 + dz_2)$,

where $a, b, c, d, f \in \mathbb{R}$, $ad + bc = 1$;

(8) $z_1' = e^{ia}z_1 + b + ci$,

$z_2' = 4e^{ia}cz_1 + i(1 - e^{2ia})z_1^2 + z_2 + d + 2c^2 i$,

where $a, b, c, d \in \mathbb{R}$.

If $\dim G_p(M^3) = 3$ and $\dim [g, g] \leq 2$, g being the Lie algebra of $G_p(M^3)$, then M^3 is locally Γ_p -equivalent (in the above sense) with one of the manifolds $\tilde{N}^3_K$ given by

(9) $(z_1 - \bar{z}_1)^2 + k(z_2 - \bar{z}_2)^2 = 1$, $0 \neq k \in \mathbb{R}$.

$G_p(\tilde{N}^3_K)$ is given by

(10) $z_1' = az_1 - kbz_2 + c$,

$z_2' = bz_1 + az_2 + d$,

where $a, b, c, d \in \mathbb{R}$. $a^2 + kb^2 = 1$.

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