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Jiří Adámek; Václav Koubek

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COEQUALIZERS IN THE GENERALIZED ALGEBRAIC CATEGORIES

Jiří ADÁMEK, Václav KOUBEK, Praha

The present paper is devoted to the study of the existence of coequalizers in a certain class of categories which is a generalization of the categories of universal algebras of a given type. This class includes also categories of topological spaces, categories of convergent spaces and the like (see [3],[6]). A generalized algebraic category is a category $A(F, G, \sigma)$ where F, G are set functors (i.e. functors from the category $\mathcal{S}$ of sets into itself), $\sigma = \{\alpha_i\}_{i \in I}$ is a type (α_i ordinals). Its objects are pairs $\langle X, \{\omega_i\}_{i \in I} \rangle$ where $\omega_i : (FX)^{\alpha_i} \longrightarrow GX$ and its morphisms from $\langle X, \{\omega_i\} \rangle$ to $\langle X', \{\omega'_i\} \rangle$ are mappings $f: X \longrightarrow X'$ such that for every $i \in I$ the diagram consisting of ω_i, Gf, ω'_i, $(Ff)^{\alpha_i}$ is commutative. Our concern will be the covariant case (both F and G covariant) and the contravariant case (both functors contravariant).

Several papers are devoted to the study of limits and colimits in the generalized algebraic categories. In the present one, we give a necessary and sufficient condition

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for the existence of coequalizers in the covariant and contravariant cases as well as the condition for their preservation by the natural forgetful functor (i.e. the functor assigning to every $\langle X , \{\omega_i\} \rangle$ its underlying set X and to every morphism its underlying mapping).

The paper has two parts: the first one contains preliminaries, the second one brings the main results.

We would like to express our gratitude to Věra Trnková for her great encouragement and help. It was she who directed our attention to the separating systems.

1. Preliminaries

Note 1.1. We say "functor" instead of covariant set functor, while "set functor" indicates that both variances are considered. We consider set functors up to natural equivalence.

We proved that every functor is naturally equivalent with an inclusions-preserving one (i.e. if $\emptyset \neq A \subset B$, then $FA \subset FB$ and for $i : A \longrightarrow B$, $i(x) = x$ we have $Fi(y) = y$) - see [7]. Thus we assume that all functors throughout this paper preserve inclusions. In particular, for arbitrary $f : X \longrightarrow Y$, $A \subset X$ we have $F(f/A) = Ff/FA$, $im\, Ff = F im f$.

Note 1.2. Let us present some functors: Q_M - the cartesian power, $C_{M,p,N}$ - the constant functor ($p : M \longrightarrow N$) which is constant to N on the category $\mathbf{S}'$ of non-void sets, $C_{M,p,N}\, \emptyset = M$, $C_{M,p,N}\, f = p$, if $dom\, f = \emptyset$. Put $C_N = C_{N, id_N, N}$. Contravariant functors: P_M - the contravariant homfunctor, $P_{M,\mathcal{F}}$ - its factorfunctor ($\mathcal{F} \subset M^M$ such that $\sim_X$ is an equivalence on M^X for every X , where $\alpha \sim_X \beta$ iff there exist $\varphi \in \mathcal{F}$ with $\varphi\alpha = \varphi\beta$): $P_{M,\mathcal{F}} X = P_M X / \sim_X$. $C^*_{0,M}$ - on $\mathcal{S}' const\, \emptyset$, $C^*_{0,M}\, \emptyset = M$.

Note 1.3. We call a functor F connected if $|F1| = 1$.

Denote $\bigvee_{i \in I} F_i$ the disjoint union of $\{F_i\}_{i \in I}$. F is connected iff whenever $F = F_1 \vee F_2$ then either $F_1 = C_\emptyset$ or $F_2 = C_\emptyset$.

If F is a contravariant functor with $F \neq C^*_{0,M}$, then always $FX \neq \emptyset$. If f is an epimorphism, then Ff is an epimorphism if F is covariant, Ff is a monomorphism if F is contravariant.

Definition 1.4. A mapping f is coarser than a mapping g iff $dom\, f = dom\, g$ and $g(x) = g(y) \Longrightarrow f(x) = f(y)$ (or, equivalently, iff there exists τ with $f = \tau \cdot g$).

Let $\{f_i\}_{i \in I}$ be a non-void collection of mappings with common domain $X \neq \emptyset$.

Co-join of $\{f_i\}$ is such an epimorphism $f = \bigcup^*_{i \in I} f_i$ that 1) all f_i are coarser than f and 2) if all f_i are coarser than a mapping g , then also f is coarser than g . If $\bigcup^* f_i = id_X$, we call $\langle X, \{f_i\}_{i \in I}\rangle$ a separating system.

Definition 1.5. A functor F preserves coequalizers (resp. separating systems), if for every $f, g : X \longrightarrow Y$

with a coequalizer h, Fh is the coequalizer of Ff, Fg (resp. for every separating system $\langle X, \{f_i\}_{i\in I}\rangle$, also $\langle FX, \{Ff_i\}_{i\in I}\rangle$ is separating). A contravariant functor F turns coequalizers into equalizers if for every $f, g : X \longrightarrow Y$ with a coequalizer h, Fh is the equalizer of Ff, Fg.

Convention 1.6. A type is 0-unary if all its elements are 0 or 1.

Let f be a mapping. $f \equiv a$ denotes that f is constant to a.

Note 1.7. A functor F preserves coequalizers iff it preserves countable unions (i.e. always $F(\bigcup_{m=1}^{\infty} X_m) = \bigcup_{m=1}^{\infty} FX_m$). See [5]. A contravariant functor turns coequalizers into equalizers iff it has the form $\coprod_{i\in I} P_{M_i}, \sigma_i$ - more precisely see [7].

Note 1.8. It is proved in [7] that the following properties of a functor F are equivalent:

1) F is connected and it preserves separating systems,
2) F is connected and it preserves co-joins (i.e. always $F(\mathcal{U}^* f_i) = \mathcal{U}^* Ff_i$),
3) for every set X and every $x, y \in FX$ there exists a mapping κ with $dom\, \kappa = X$ and $F\kappa(x) = F\kappa(y)$ such that every mapping with the same property is coarser than κ.

Note 1.9. Denote $A(F, G, 1) = A(F, G, \{1\})$. It is easy to verify that $A(F, G, \sigma)$ with $\sigma = \{M_i\}_{i\in I}$ is isomorphic to $A((\bigvee_{i\in I} Q_{M_i}) . F, G, 1)$.

Clearly, for an arbitrary functor F, $\bigvee_{i\in I} Q_{M_i} \cdot F$ preserves coequalizers iff F preserves them and $|M_i| \le 1$ or F is constant, M_i arbitrary.

Note 1.10. Let k be the coequalizer of $\varphi, \psi : M \longrightarrow N$ in $\mathcal{S}$. Then $k^{-1}k(A) = A \Rightarrow \varphi^{-1}(A) = \psi^{-1}(A)$.

Proof. $t \in \varphi^{-1}(A)$, $k\varphi = k\psi \Longrightarrow \psi(t) \in k^{-1}k(A) = A \Longrightarrow t \in \psi^{-1}(A)$.
Analogously, $t \in \psi^{-1}(A) \Longrightarrow t \in \varphi^{-1}(A)$.

Proposition 1.11. Let F be a functor not preserving coequalizers. Then for every set Z there exist mappings $f, g : X \longrightarrow Y$ such that $Z \subset X \cap Y$, $f/Z = g/Z$ is the inclusion, and there exist disjoint $A, B \subset FY$ such that if we denote k the coequalizer of f and g, k^* an epimorphism from Y, k^*/Z -constant, $k^*/Y - Z \equiv k/Y - Z$, we have

1) $Fk A \cap Fk B \neq \emptyset$,

2) $x \in A \cup B$, $Fk^*(x) = Fk^*(y) \Longrightarrow y \in A \cup B$,

3) $(Ff)^{-1}A = (Fg)^{-1}A$, $(Ff)^{-1}B = (Fg)^{-1}B$.

To prove this proposition we shall need some special facts concerning set functors, which we present separately.

Definition A. Define a functor $\underline{F}$; $\underline{F}X = \{\tilde{t}, \tilde{t} = \exp X$ or $\tilde{t}$ is a filter on $X\}$, for $f : X \longrightarrow Y$, $\underline{F}f(\tilde{t}) = \{B; B \subset Y, f(A) \subset B$ for some $A \in \tilde{t}\}$.
Let F be a functor. Define $\mathcal{F}_F^X : FX \longrightarrow \underline{F}X$ for

every set X : $\mathcal{F}_F^X(x) = \{A ; A \subset X, x \in FA\}$.

Statement B. Let F be a functor, $f: X \longrightarrow Y$ a monomorphism, $x \in FX$. Then $\mathcal{F}_F^Y(Ff(x)) = Ff(\mathcal{F}_F^*(x))$.

Proof. See [1].

Statement C. A functor F preserves coeuqualizers iff for every $x \in FX$, $\mathcal{F}_F^X(x)$ is an ultrafilter closed under countable intersections or $exp\, X$.

Proof. See [5].

Statement D. A filter $\tilde{t}$ on X is an ultrafilter closed under countable intersections iff for every countable disjoint decomposition of X, $X = \bigcup_{i=1}^{\infty} X_i$, there exists m with $X_m \in \tilde{t}$.

Proof of Proposition 1.11. If F does not preserve coequalizers, there exists $\xi \in FY$ such that $\mathcal{F}_F^Y(\xi)$ is neither $exp\, Y$ nor an ultrafilter closed under countable intersection and so there exists a disjoint decomposition of Y, $Y = \bigcup_{i=1}^{\infty} Y_i$ with $Y_n \notin \mathcal{F}_F^Y(\xi)$ for every $n = 1, 2, \ldots$. Let $\mathcal{Z}$ be the set of all integers and put $X_1 = \mathcal{Z} \times Y$. Without loss of generality $X_1 \cap Z = \emptyset$. Put $X = X_1 \cup Z$. Put $f, g : X \longrightarrow X$, $f = id_X$, $g/_Z = id/_Z$, $g(\langle k, y\rangle) = \langle k+1, y\rangle$. Let $\ell, h : Y \longrightarrow X$, $\ell(y) = \langle 0, y\rangle$, $h(y) = \langle n, y\rangle \Longleftrightarrow y \in Y_n$. Notice that f, g, ℓ, h are monomorphisms and thus they fulfil the assumptions of the statement B. Further, g is an automorphism and so g^k, $k \in \mathcal{Z}$ has sense. As $Ff = id$ and Fg is an automorphism, $K(t) = K(u) \Longleftrightarrow \exists m \in \mathcal{Z}$, $t = (Fg)^n(u)$, where K is the coequalizer of Ff and Fg. Let us prove: $K(Fh(\xi)) \neq K(F\ell(\xi))$. Assume that, on the contrary, $Fh(\xi) = (Fg)^n (F\ell(\xi))$ for some $n \in \in \mathbb{Z}$. Then

$$g^n \ell(Y) \in F(g^n \ell)(\mathcal{F}_F^Y(\xi)) = \mathcal{F}_F^X(Fg^n \ell(\xi)) =$$
$$= \mathcal{F}_F^X(Fh(\xi)) = Fh(\mathcal{F}_F^Y(\xi))$$

and so there exists $H \in \mathcal{F}_F^Y(\xi)$ with $h(H) \subset g^n \ell(Y)$. But then $H \subset Y_m$ (see the definition of h, ℓ and g) and that is a contradiction as $H \in \mathcal{F}_F^Y(\xi)$, $Y_m \supset H \Rightarrow \Rightarrow Y_m \in \mathcal{F}_F^Y(\xi)$. Therefore $K(F\ell(\xi)) \neq K(Fh(\xi))$. Let k, k^* be as in the proposition. Evidently, k/X_1 is the projection $\mathbb{Z} \times Y \longrightarrow Y$ and so $k\ell = kh$ and $Fk(F\ell(\xi)) = Fk(Fh(\xi))$. Now put $A = K^{-1}K(F\ell(\xi))$, $B = (Fk^*)^{-1}(Fk^*(F\ell(\xi)) - A$. Evidently, $A \cap B = \emptyset$.

1) $F\ell(\xi) \in A$, $Fh(\xi) \in B$ and $Fk(F\ell(\xi)) = Fk(Fh(\xi))$,

2) $x \in A \cup B$, $Fk^*(x) = Fk^*(y) \Rightarrow y \in A \cup B$,

3) as $Fk^* Ff = Fk^* Fg$, Fk^* is coarser than K and so $K^{-1}(K(B)) = B$. Then, $(Ff)^{-1}A = (Fg)^{-1}A$, $(Ff)^{-1}B = (Fg)^{-1}B$ (see 1.10).

<u>Lemma 1.12</u>. Let $\mathcal{A}(F, C_{M\,n\,N}, 1)$ have coequalizers. Then either F preserves coequalizers or $|N| \leq 1$.

<u>Proof</u>. Let neither hold. Then there exist $f.g: X \rightarrow Y$ with a coequalizer k such that there is $A \subset FY$ with

$(Ff)^{-1}A = (Fg)^{-1}A = \mathcal{A}$ and there exist $a \in A$, $b \in FY - A$, $Fh(a) = Fh(b)$ (see Lemma 2.2). (Necessarily $X \neq \emptyset \neq Y$.) Let $t, u \in N$, $t \neq u$, let $\omega^X : FX \longrightarrow N$, $\omega^X/\mathcal{A} \equiv t$, $\omega^X/FX - \mathcal{A} \equiv u$; let $\omega^Y : FY \longrightarrow N$, $\omega^Y/A \equiv t$, $\omega^Y/FY - A \equiv u$. Obviously $f, g : \langle X, \omega^X \rangle \longrightarrow \langle Y, \omega^Y \rangle$ and this couple does not have a coequalizer in $A(F, C_{M \pi N}, 1)$. Indeed, let $\ell : \langle Y, \omega^Y \rangle \longrightarrow \langle V, \omega^V \rangle$ and $\ell f = \ell g$. Then ℓ is coarser than h and so $F\ell(a) = F\ell(b)$. But $\omega^V F\ell(a) = \omega^Y(a) = t \neq u = \omega^Y(b) = \omega^V F\ell(b)$ which is a contradiction.

2. Main results.

The covariant case. Theorem 2.1. $A(F, G, \sigma)$ with F, G covariant has coequalizers preserved by the forgetful functor iff

either F preserves coequalizers and σ is 0-unary,

or F is constant,

or $G = C_{M \pi N}$ with $|N| \leq 1$.

Proof. I. Let $A(F, G, \sigma)$ have such coequalizers. With the help of 1.9 it suffices to prove that if, moreover, $\sigma = \{1\}$ and $G \neq C_{M \pi 1}$, C_0, then F preserves coequalizers. Assume the contrary. Evidently there exists $Z \neq \emptyset$ with $|GZ| > 1$, let $a, b \in GZ$, $a \neq b$. Let us use the proposition 1.11 to F, Z. Thus we have

$f, g : X \longrightarrow Y$ with $Z \subset X \cap Y$, $f/_Z = g/_Z = {id}/_Z$ and $A \subset FY$, $Fk(A) \cap Fk(FY - A) = \emptyset$ (where k is the coequalizer of f and g in $\mathcal{S}$), $(Ff)^{-1}A = (Fg)^{-1}A = \{a\}$. Let $\alpha \in A$, $\beta \in FY - A$, $Fk(\alpha) = Fk(\beta)$. Put $\omega^X : FX \longrightarrow GX$, $\omega^X/a \equiv a$, $\omega^X/FX - a \equiv b$, $\omega^Y : FY \longrightarrow GY$, $\omega^Y/_A \equiv Gf(a)$, $\omega^Y/_{FY-A} \equiv Gf(b)$.

Evidently $f, g : \langle X, \omega^X \rangle \longrightarrow \langle Y, \omega^Y \rangle$. Let $k : \langle Y, \omega^Y \rangle \longrightarrow \langle V, \omega^V \rangle$ be a coequalizer of f and g in $A(F, G, 1)$. Clearly, $k/_Z$ is a monomorphism and so $Gk/_{GZ}$ is a monomorphism, $Gk(Gf(a)) \neq Gk(Gf(b))$. But $Gk(Gf(a)) = Gk(\omega^Y(\alpha)) = \omega^V Fk(\alpha) = \omega^V Fk(\beta) = Gk\, \omega^Y(\beta) = Gk(Gf(b))$ which is a contradiction.

II. Evidently $A(F, C_{M \cap N}, \sigma)$ with $|N| \leq 1$ has coequalizers preserved by the forgetful functor. Now it suffices to prove that if F preserves coequalizers, $A(F, G, 1)$ has such coequalizers. Let $f, g : \langle X, \omega^X \rangle \longrightarrow \langle Y, \omega^Y \rangle$, let $k : Y \longrightarrow Z$ be a coequalizer of f and g in $\mathcal{S}$. As Fk is the coequalizer of Ff and Fg and as $(Gk \circ \omega^Y \circ) Ff = Gk \circ Gf \circ \omega^X = Gk \circ Gg \circ \omega^X = (Gk\omega^Y) \circ Fg$, there exists ω with $Gk \circ \omega^Y = \omega \circ Fk$. Thus $k : \langle Y, \omega^Y \rangle \longrightarrow \langle Z, \omega \rangle$. Let us prove that this is the coequalizer in $A(F, G, 1)$.

Let $\ell : \langle Y, \omega^{Y} \rangle \longrightarrow \langle V, \omega^{V} \rangle$ be arbitrary with $\ell f = \ell g$. There exists a unique τ with $\ell = \tau k$. Let us show that $\tau : \langle Z, \omega \rangle \longrightarrow \langle V, \omega^{V} \rangle$. We have
$\omega^{V}. F\tau . Fk = \omega^{V}. F\ell = G\ell . \omega^{Y} = G\tau . Gk . \omega^{Y} = G\tau . \omega . Fk$
and as Fk is an epimorphism, there follows
$\omega^{V} \cdot F\tau = G\tau \cdot \omega$.

<u>Theorem 2.2</u>. $A(F, G, \sigma)$, with F and G covariant, has coequalizers iff either F preserves coequalizers and σ is 0-unary,
or F is constant,
or G is connected and it preserves separating systems.

<u>Proof</u>. Assume $\sigma = \{1\}$.

I) Let $A(F, G, 1)$ have coequalizers. Let F not preserve coequalizers. We shall show that G has the property 3) from 1.8. We assume that G is not constant because otherwise (due to 1.12) it is connected, preserving separating systems.

a) To prove 3) we shall, for arbitrary Z; $x_1, x_2 \in GZ$, construct $f, g : \langle X, \omega^{X} \rangle \longrightarrow \langle Y, \omega^{Y} \rangle$ with $Z \subset Y$ and such that: if $\varphi : Z \longrightarrow T$, then
$(*)$ $G\varphi(x_1) = G\varphi(x_2)$ iff there exists
$\tilde{\varphi} : \langle Y, \omega^{Y} \rangle \longrightarrow \langle T', \omega^{T'} \rangle$ such that $\tilde{\varphi}/_Z = \varphi$ and
$\tilde{\varphi} f = \tilde{\varphi} g$. That will be sufficient because then if
$\ell : \langle Y, \omega^{Y} \rangle \longrightarrow \langle Y, \omega \rangle$ is the coequalizer of f and g in $A(F, G, 1)$, then κ - the domain-range restriction of ℓ to Z and $\ell(Z)$ is the mapping from 3):

first $G\varkappa(x_1) = G\varkappa(x_2)$ (as there exists $\tilde{\varkappa} = \ell$), second, if $G\varkappa'(x_1) = G\varkappa'(x_2)$, then there exists $\tilde{\varkappa}'$ and as ℓ is the coequalizer, there exists τ with $\tilde{\varkappa}' = \tau\ell$ and so $\varkappa' = [\tau/\varkappa(x)]\cdot\varkappa\cdot\ell$. Now let us construct f, g. As G is not constant, there exists Z', $GZ \subsetneqq GZ'$; let $z \in GZ' - GZ$. Let us use the proposition 1.11 on F, Z'. We have $f, g : X \longrightarrow Y$; $A, B \subset Y$ such that (k, k^* as in 1.11)

$Z' \subset X \cap Y$, $f/Z' = g/Z'$ is the inclusion,

$A \cap B = \emptyset$, $Fk(A) \cap Fk(B) \neq \emptyset$, $(Fk^*)^{-1} Fk^* A = A$,

$(Fk^*)^{-1} Fk^* B = B$, $(Ff)^{-1}A = (Fg)^{-1}A = \mathcal{A}$, $(Ff)^{-1}B = (Fg)^{-1}B = \mathcal{B}$.

Let $x_1', x_2', z' \in GZ'$, $Gf(x_i') = Gg(x_i') = x_i$, $i = 1, 2$; $Gf(z') = Gg(z') = z$.

Let us define

$\omega^X : FX \longrightarrow GX$, $\omega^X/\mathcal{A} \equiv x_1'$, $\omega^X/\mathcal{B} \equiv x_2'$, $\omega^X/F\text{-}X\text{-}(\mathcal{A} \cup \mathcal{B}) \equiv z'$;

$\omega^Y : FY \longrightarrow GY$, $\omega^Y/A \equiv x_1$, $\omega^Y/B \equiv x_2$, $\omega^Y/FY - (A \cup B) \equiv z$.

Clearly $f, g : \langle X, \omega^X \rangle \longrightarrow \langle Y, \omega^Y \rangle$. Let us prove (*).

1) Let $\varphi : Z \longrightarrow T$, $G\varphi(x_1) = G\varphi(x_2)$. Let $\tilde{\varphi} : Y \xrightarrow{onto} T$, $\tilde{\varphi}/Z = \varphi$, $\tilde{\varphi}/Y - Z = k/Y - Z$. Let us show that $G\tilde{\varphi} \circ \omega^Y$ is coarser than $F\tilde{\varphi}$. Clearly, k^* is coarser than $\tilde{\varphi}$. Therefore $F\tilde{\varphi}(t) = F\tilde{\varphi}(u) \Longrightarrow$ $\Longrightarrow Fk^*(t) = Fk^*(u) \Longrightarrow$ either both t and u are elements of $A \cup B$ or neither of them $\Longrightarrow$ either $\omega(t), \omega(u) \in GZ$ and so $G\varphi \circ \omega(t) = G\varphi \circ \omega(u)$ or $\omega(t) = \omega(u)$. Therefore $G\tilde{\varphi} \circ \omega^Y$ is really

coarser than $F\tilde{\varphi}$ and we may define $\omega^{\tilde{T}}$ by $\omega^{\tilde{T}}(F\tilde{\varphi}(t)) = G\tilde{\varphi}(\omega(t))$. Clearly $\tilde{\varphi}: \langle X, \omega^Y\rangle \longrightarrow \langle \tilde{T}, \omega^{\tilde{T}}\rangle$, $\tilde{\varphi}f = \tilde{\varphi}g$.

2) Let $\tilde{\varphi}: \langle Y, \omega^Y\rangle \longrightarrow \langle T_1, \omega^{T_1}\rangle$, $\tilde{\varphi}f = \tilde{\varphi}g$. Then $\tilde{\varphi}$ is coarser than k and as $Fk(a) = Fk(b)$ for some $a \in A$, $b \in B$, we have $F\tilde{\varphi}(a) = F\tilde{\varphi}(b)$. Therefore $G\tilde{\varphi}(x_1) = G\tilde{\varphi}(\omega^Y(a)) = \omega^{\tilde{T}}F\tilde{\varphi}(a) = \omega^{\tilde{T}}F\tilde{\varphi}(b) = G\tilde{\varphi}(\omega^Y(b)) = G\tilde{\varphi}(x_2)$. Therefore if $\varphi = \tilde{\varphi}/Z$, then $G\varphi(x_1) = G\varphi(x_2)$.

II) Due to Theorem 2.1 it is sufficient to show that if G preserves separating systems and it is connected, then $A(F, G, 1)$ has coequalizers:

Let $f, g: \langle X, \omega^X\rangle \longrightarrow \langle Y, \omega^Y\rangle$, let $\mathfrak{M} = \{\beta;\ \beta: Y \longrightarrow Y',\ \beta f = \beta g$ and $G\beta \circ \omega^Y$ coarser than $G\beta\}$, $\mathfrak{M} \neq \emptyset$, as $\beta \in \mathfrak{M}$ if β is constant (as then $G\beta$ is constant while G is connected). Let $\ell = \bigcup^* \mathfrak{M}$, $\ell: Y \longrightarrow V$, then $G\ell = \bigcup^*_{\beta \in \mathfrak{M}} G\beta$ (see 1.8). Therefore $F\ell(t) = F\ell(u) \Longrightarrow F\beta(t) = F\beta(u)$ for each $\beta \in \mathfrak{M} \Longrightarrow G\beta\omega^Y(t) = G\beta\omega^Y(u)$ for each $\beta \in \mathfrak{M} \Longrightarrow G\ell\omega^Y(t) = G\ell\omega^Y(u)$ and we may define $\omega^V: FV \longrightarrow GV$ by $\omega^V(F\ell(t)) = G\ell(\omega^Y(t))$. Let us show that $\ell: \langle Y, \omega^Y\rangle \longrightarrow \langle V, \omega^V\rangle$ is the coequalizer of f and g in $A(F, G, 1)$. If $\ell': \langle Y, \omega^Y\rangle \longrightarrow \langle V', \omega^{V'}\rangle$, $\ell' f = \ell' g$, then $G\ell' \circ \omega^Y$ $(= \omega^{V'}F\ell')$ is coarser than $F\ell'$ and so $\ell' \in \mathfrak{M}$. Therefore there exists a unique τ with $\ell' = \tau \ell$. Let us prove that $\tau : \langle V, \omega^V \rangle \longrightarrow \langle V', \omega^{V'} \rangle$.

$$\omega^{V'} . F\tau . F\ell = \omega^{V'} . F\ell' = G\ell' . \omega^Y = G\tau . G\ell . \omega^Y = G\tau . \omega^V F\ell$$

and as ℓ (and $F\ell$) is an epimorphism, we have $\omega^{V'} . F\tau = G\tau . \omega^V$

<u>The contravariant case</u>. <u>Theorem 2.3.</u> Let F, G be contravariant functors. The following statements are equivalent:

1) $A(F, G, \sigma)$ has coequalizers,

2) $A(F, G, \sigma)$ has coequalizers preserved by the forgetful functor,

3) either G turns coequalizers into equalizers, or $F = C^*_{0,M}$.

<u>Proof</u>. Evidently it suffices to prove the theorem for $\sigma = \{1\}$. $1) \Longrightarrow 3)$. Let $F \neq C^*_{0,M}$ and let G not turn coequalizers into equalizers. Let $f, g : X \longrightarrow Y$ with a coequalizer $k : Y \longrightarrow Z$ be such that Gk is not the equalizer of Gf and Gg . As $kf = kg$, we have $Gf\, Gk = Gg\, Gk$ and $Im\, Gk \subset \{t \in GY, Gf(t) = Gg(t)\}$. As clearly these sets are not equal, there exists $t \in GY - Im\, Gk$ with $Gf(t) = Gg(t) = u$. Clearly $X \neq \emptyset$, so $FX \neq \emptyset$. Let $\omega^X : FX \longrightarrow GX$ be the constant onto u , $\omega^Y : FY \longrightarrow GY$ the constant onto t . Clearly $f, g : \langle X, \omega^X \rangle \longrightarrow \langle Y, \omega^Y \rangle$. Let $\ell : \langle Y, \omega^Y \rangle \longrightarrow \langle Z, \omega^Z \rangle$ be their coequalizer in $A(F, G, 1)$. Clearly $Z \neq \emptyset$ and so $FZ \neq \emptyset$, let

$x \in FZ$, $t = \omega^Y F\ell(x) = G\ell(\omega^Z(x))$ and so $t \in Im\, G\ell$. But as $\ell f = \ell g$, ℓ is coarser than k, $\ell = \tau k$ and so $G\ell = Gk \circ G\tau$ from which it immediately follows that $Im\, G\ell \subset Im\, Gk$ and that is a contradiction.

3) $\Longrightarrow$ 2). The case that $F = C^*_{0,M}$ is trivial. If G turns coequalizers into equalizers, we proceed analogously as in 2.1.

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Matematicko-fyzikální fakulta
Karlova universita,
Praha 8,Sokolovská 83, Czechoslovakia

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