Commentationes Mathematicae Universitatis Carolinae

Ladislav Nebeský

Planar permutation graphs of paths

Commentationes Mathematicae Universitatis Carolinae, Vol. 13 (1972), No. 2, 253--256

Persistent URL: http://dml.cz/dmlcz/105413

Commentationes Mathematicae Universitatis Carolinae
13,2 (1972)

PLANAR PERMUTATION GRAPHS OF PATHS

Ladislav NEBESKÝ, Praha

The present note gives the solution of the first of three problems stated in Chartrand and Frechen [2]; formerly, this problem appears in an implicite form in Chartrand and Harary [1]. (This problem has a certain relation to the question concerning mathematical linguistics discussed in [3].)

Let $m \geq 1$. Consider a path A_m with the set of vertices $R = \{r_1, \ldots, r_{m+1}\}$ and the set of edges $E_R = \{r_1 r_2, \ldots, r_m r_{m+1}\}$. By B_m we shall denote a disjoint copy of the path A_m such that B_m has the set of vertices $S = \{s_1, \ldots, s_{m+1}\}$ and the set of edges $E_S = \{s_1 s_2, \ldots, s_m s_{m+1}\}$. Let α be a permutation on the set $\{1, \ldots, m+1\}$. By $P_\alpha(A_m)$ we denote the graph with the set of vertices $R \cup S$ and the set of edges $E_R \cup E_S \cup \{r_1 s_{\alpha(1)}, \ldots, r_{m+1} s_{\alpha(m+1)}\}$. The graph $P_\alpha(A_m)$ is a special case of permutation graphs which were studied in [1] and [2].

Integers will be denoted by e, f, g, h, i, j and k. We shall write $med(f, g, h)$ if either $f < g < h$

AMS, Primary: 05C99 Ref. Ž. 8.83

or $h < g < f$.

<u>Theorem</u>. A necessary and sufficient condition for $P_\alpha(A_n)$ to be planar, is that for any i, j, k such that $1 < i < j < k < n+1$, at most one of the following two statements hold:

(1) $$med(\alpha(i), \alpha(j), \alpha(i-1)) ,$$

(2) $$med(\alpha(k), \alpha(j), \alpha(k+1)) .$$

<u>Proof</u>. Necessity: Assume that $P_\alpha(A_n)$ is planar and that there exist i, j, k such that $1 < i < j < k \le n$ and both (1) and (2) hold. Let e, f, g, h be such that $\{e, f, g, h\} = \{i, i-1, k, k+1\}$ and $\alpha(e) < \alpha(f) < \alpha(j) < \alpha(g) < \alpha(h)$. By G we denote the subgraph of $P_\alpha(A_n)$ consisting of the path between r_{i-1} and r_{k+1} in A_n, the path between $s_{\alpha(e)}$ and $s_{\alpha(h)}$ in B_n, and the edges $r_e s_{\alpha(e)}$, $r_f s_{\alpha(f)}$, $r_j s_{\alpha(j)}$, $r_g s_{\alpha(g)}$, $r_h s_{\alpha(h)}$. Obviously, G is homeomorphic to the complete bipartite graph $K_{3,3}$; the vertices r_j, $s_{\alpha(f)}$, $s_{\alpha(g)}$ and r_i, r_k, $s_{\alpha(j)}$ represent its two levels. Thus $P_\alpha(A_n)$ is not planar, which is a contradiction.

Sufficiency: Consider a cartesian plane. For every j, $1 \le j \le n+1$, we define the points $v_j = (j, \alpha(j))$, $w_j = (0, j)$ and $z_j = (n+2, j)$. We shall say that a point v_j is of the first or the second kind if there exist k, $1 \le k \le n$, such that the intervals $v_j z_{\alpha(j)}$ and $v_k v_{k+1}$ or the intervals $v_j w_{\alpha(j)}$ and $v_k v_{k+1}$, respectively, cross. It is readily seen that no point v_j is simultaneously of the first and of the second kind. We shall say that a point v_j is of the third kind if it is neither of the first nor of the second kind. The graph $P_\alpha(A_n)$ can be embedded in the plane as follows: every vertex r_e is drawn as the point v_e ; every vertex s_f is drawn as the point w_f ; every edge $r_g r_{g+1}$ as the interval $v_g v_{g+1}$; every edge $s_h s_{h+1}$ as the interval $w_h w_{h+1}$; every edge $r_i s_{\alpha(i)}$ as the interval $v_i w_{\alpha(i)}$, when v_i is of the first or the third kind and as a suitable arc passing through the point $x_{\alpha(i)}$, when v_i is of the second kind. Obviously, there are arcs C_j connecting w_j with x_j such that no two of them intersect and that C_j meets the oblong $\langle 0, \dots, n+2 \rangle \times \langle 1, \dots, n+1 \rangle$ only in w_j and x_j. Thus, it suffices to extend the intervals $v_i x_{\alpha(i)}$ for v_i of the second kind by $C_{\alpha(i)}$.

Acknowledgment. I would like to express my thanks to A. Pultr for his advice which improved the formulations in this paper.

References

[1] G. CHARTRAND and F. HARARY: Planar permutation graphs, Ann.Inst.H.Poincaré,Sect.B3(1967),433-438.

[2] G. CHARTRAND and J.B. FRECHEN: On the chromatic number of permutation graphs, in:Proof Techniques in Graph Theory(Ed.F.Harary),Academic Press,New York and London 1969, pp.21-24.

[3] L. NEBESKÝ: A planar test of linguistic projectivity (to appear in Kybernetika).

Filosofická fakulta
Karlova universita
nám.Krasnoarmějců 2
Praha 1
Československo

(Oblatum 17.11.1971)