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CONCERNING RESOLVENT KERNELS OF VOLTERRA INTEGRAL EQUATIONS

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In this paper, a class of linear Volterra integral operators of convolution type is being investigated such that the kernel of the operator satisfies a certain linear ordinary differential equation with constant coefficients. It is shown that for every such operator there exists a linear ordinary differential equation, describing in some sense the properties of the operator. The latter differential equation makes it possible to compute effectively resolvent kernels of Volterra integral equations.

1. Notation. Let C denote the set of all complex numbers. Let R_+ denote the set of all non-negative real numbers. We shall denote by $\mathcal{C}$ the set of all continuous functions $f: R_+ \to C$ and by $\mathcal{C}^{(k)}$ (for k positive integer) the set of all k-times continuously differentiable functions $f: R_+ \to C$. Sometimes we write $\mathcal{C}^{(0)}$ instead of $\mathcal{C}$. If n, k are integers, $0 \le n \le k$, and $f \in \mathcal{C}^{(k)}$ then the symbol $f^{(n)}$ denotes the n-th derivative of the function f. Especially, $f^{(0)}$ denotes the function f itself.

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Secondary -

2. <u>Definition</u>. Let a, b be two continuous functions. A linear integral operator $T: \mathcal{C} \to \mathcal{C}$ is defined as follows:

$$(1) \qquad Tx(t) = a(t) + \int_0^t b(t-s)\,x(s)\,ds .$$

3. <u>Remark</u>. The substitution $u = t - s$ in the integral $\int_0^t b(t-s)\,x(s)\,ds$ gives

$$(2) \qquad \int_0^t b(t-s)\,x(s)\,ds = \int_0^t x(t-u)\,b(u)\,du$$

and the relation (1) becomes

$$(3) \qquad Tx(t) = a(t) + \int_0^t x(t-s)\,b(s)\,ds .$$

4. <u>Lemma</u>. For any non-negative integer k and for any given functions $a, b \in \mathcal{C}^{(k)}$, the operator T maps $\mathcal{C}^{(k-1)}$ into $\mathcal{C}^{(k)}$. Moreover, for every $u \in \mathcal{C}^{(k-1)}$ and its image $v = Tu$,

$$(4) \qquad v(t) = a(t) + \int_0^t b(t-s)\,u(s)\,ds ,$$

the following is true:

$$(5.k) \qquad v^{(k)}(t) = \sum_{j=0}^{k-1} b^{(k-j-1)}(0)\,u^{(j)}(t) + a^{(k)}(t) + \int_0^t b^{(k)}(t-s)\,u(s)\,ds .$$

<u>Proof</u> (by induction). The theorem on differentiation of an integral with respect to a parameter ensures that the function v given by (4) is differentiable if a, $b \in \mathcal{C}^{(1)}$, $u \in \mathcal{C}$. The derivative of v is then

$$v^{(1)}(t) = b(0)u(t) + a^{(1)}(t) + \int_0^t b^{(1)}(t-s)u(s)\,ds .$$

Thus the operator T maps $\mathcal{C}^{(0)}$ into $\mathcal{C}^{(1)}$; hence (5.1) holds. Now, let us suppose that

$$(5.k-1) \qquad v^{(k-1)}(t) = \sum_{j=0}^{k-2} b^{(k-j-2)}(0)u^{(j)}(t) + a^{(k-1)}(t) + \int_0^t b^{(k-1)}(t-s)u(s)\,ds$$

holds and furthermore $a, b \in \mathcal{C}^{(k)}$, $u \in \mathcal{C}^{(k-1)}$. Then the function $v^{(k-1)}$ is continuously differentiable and the differentiation with respect to t on both sides of (5.k-1) gives

$$v^{(k)}(t) = \sum_{j=0}^{k-2} b^{(k-j-2)}(0)u^{(j+1)}(t) + b^{(k-1)}(0)u(t) + a^{(k)}(t) + \int_0^t b^{(k)}(t-s)u(s)\,ds .$$

Substituting $j+1 \longrightarrow j$ in the last equation (5.k) is easily obtained.

5. Remark. From (5.k) and (2) it follows immediately

$$(6.k) \qquad v^{(k)}(t) = \sum_{j=0}^{k-1} b^{(k-j-1)}(0)u^{(j)}(t) + a^{(k)}(t) + \int_0^t u(t-s)\,b^{(k)}(s)\,ds .$$

Supposing now $a, u \in \mathcal{C}^{(k)}$, $b \in \mathcal{C}^{(k-1)}$, and using (2) to modify (4) to the form

$$(7) \qquad v(t) = a(t) + \int_0^t u(t-s)\,b(s)\,ds ,$$

we obtain from Lemma 4:

$$(8.k)\qquad v^{(k)}(t) = \sum_{j=0}^{k-1} u^{(k-j-1)}(0)\, b^{(j)}(t) + a^{(k)}(t) + \\ + \int_0^t u^{(k)}(t-s)\, b(s)\, ds .$$

This may also be written, using the relation (2), as

$$(9.k)\qquad v^{(k)}(t) = \sum_{j=0}^{k-1} u^{(k-j-1)}(0)\, b^{(j)}(t) + a^{(k)}(t) + \\ + \int_0^t b(t-s)\, u^{(k)}(s)\, ds .$$

Relations (5.k),(6.k),(8.k) and (9.k) make it possible for the operator T to be conveniently characterized by certain linear differential operators.

6. Theorem. Let $A_0, A_1, \dots, A_m$ be complex constants and $a, b \in \mathcal{C}^{(m)}$. Let b be the solution of the initial value problem

$$(10)\ \sum_{k=0}^{m} A_k x^{(k)} = 0,\ x^{(k)}(0) = b^{(k)}(0) = b_k,\ k = 0, 1, \dots, m-1 .$$

If $u \in \mathcal{C}^{(m-1)}$, then the function $v = Tu$ is the solution of the initial value problem

$$(11)\qquad \sum_{k=0}^{m} A_k y^{(k)} = \sum_{k=0}^{m-1} B_k u^{(k)}(t) + \sum_{k=0}^{m} A_k a^{(k)}(t) ,$$

where

$$B_k = \sum_{j=k+1}^{m} A_j b_{j-k-1}$$

with the initial conditions

$$(12)\ v(0) = a(0),\ v^{(j)}(0) = b_{j-1} u(0) + b_{j-2} u^{(1)}(0) + \dots + \\ + b_0 u^{(j-1)}(0) + a^{(j)}(0),\quad j = 1, 2, \dots, m-1 .$$

__Proof__. Let

$$v(t) = a(t) + {}_0\int^t b(t-s)\,u(s)\,ds .$$

Then, according to (6.k), there holds for all integers

$0 < k \leq m$

$$A_k v^{(k)}(t) = A_k \sum_{j=0}^{k-1} b_{k-j-1} u^{(j)}(t) + A_k a^{(k)}(t) + {}_0\int^t u(t-s) A_k b^{(k)}(s)\,ds .$$

Hence, summing over all k's from 0 to m and using the assumption $\sum_{k=0}^{m} A_k b^{(k)}(s) = 0$, we have

$$\sum_{k=0}^{m} A_k v^{(k)}(t) = \sum_{k=0}^{m} A_k \sum_{j=0}^{k-1} b_{k-j-1} u^{(j)}(t) + \sum_{k=0}^{m} A_k a^{(k)}(t) .$$

From the Dirichlet's formula for double sums we obtain

$$\sum_{k=0}^{m} A_k v^{(k)}(t) = \sum_{j=0}^{m-1} u^{(j)}(t) \sum_{k=j+1}^{m} A_k b_{k-j-1} + \sum_{k=0}^{m} A_k a^{(k)}(t) ,$$

which is equivalent to (11).

7. __Remark__. If the function a also satisfies (10), then (11) becomes

$$(13)\quad \sum_{k=0}^{m} A_k y^{(k)} = \sum_{k=0}^{m-1} B_k u^{(k)}(t);\quad B_k = \sum_{j=k+1}^{m} A_j b_{j-k-1} .$$

Let Theorem 6 be illustrated by two simple examples.

8. __Examples__. 1. Let $a \in \mathscr{C}^{(2)}$ be arbitrary and

$$b(t) = \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t}, \quad \beta_1, \beta_2, \lambda_1, \lambda_2 \in C .$$

Then $b_0 = \beta_1 + \beta_2$, $b_1 = \lambda_1 \beta_1 + \lambda_2 \beta_2$, $A_0 = \lambda_1 \lambda_2$,

$$A_1 = -(\lambda_1 + \lambda_2),\ A_2 = 1,\ B_0 = -(\beta_1 \lambda_2 + \beta_2 \lambda_1),\ B_1 = \beta_1 + \beta_2 .$$

For any $x \in \mathscr{C}^{(1)}$ the function

$$y(t) = a(t) + {}_0\int^t (\beta_1 e^{\lambda_1 (t-s)} + \beta_2 e^{\lambda_2 (t-s)})\,x(s)\,ds$$

solves the following initial value problem

$$y^{(2)} - (\lambda_1 + \lambda_2) y^{(1)} + \lambda_1 \lambda_2 y = \lambda_1 \lambda_2 a(t) - (\lambda_1 + \lambda_2) a^{(1)}(t) +$$

$+ a^{(2)}(t) - (\beta_1 \lambda_2 + \beta_2 \lambda_1) x(t) + (\beta_1 + \beta_2) x^{(1)}(t)$,

with the initial conditions

$$y(0) = a(0), \ y^{(1)}(0) = a^{(1)}(0) + (\beta_1 + \beta_2) x(0) \ .$$

2. Let $a(t) = \sum_{i=0}^{m-1} a_i t^i$ be an arbitrary polynomial and $b(t) = t^{m-1}$.

Then $b_j = 0$ for $j = 0, 1, \ldots, m-2$, $b_{m-1} = (m-1)!$, $A_j = 0$ for $j = 0, 1, \ldots, m-1$, $A_m = 1$, $B_0 = (m-1)!$, $B_j = 0$ for $j = 1, 2, \ldots, m-1$.

For any $x \in \mathcal{C}^{(m-1)}$ the function

$$y(t) = a(t) + \int_0^t (t-s)^{m-1} x(s)\, ds$$

solves the following initial value problem

$$y^{(m)} = (m-1)!\, x(t), \ y^{(j)}(0) = a^{(j)}(0) = j!\, a_j, \ j = 0, 1, \ldots, m-1.$$

9. <u>Remark</u>. Theorem 6 may serve as a useful tool for the computation of fixed points of the integral operator (1) or, which amounts to the same, for the solution of a Volterra integral equation of the second kind. Actually, a function $x \in \mathcal{C}^{(m)}$ is a fixed point of the operator T iff x is the solution of the equation

$$x(t) = a(t) + \int_0^t b(t-s)\, x(s)\, ds, \quad t \geq 0 \ .$$

It then follows from Theorem 6 that x solves the initial value problem

(14) $$\sum_{k=0}^{m} C_k x^{(k)} = \sum_{k=0}^{m} A_k a^{(k)}(t),$$

$$C_k = A_k - \sum_{j=k+1}^{m} A_j b_{j-k-1}, \ k = 0, 1, \ldots, m-1, \ C_m = A_m \ ,$$

with the initial conditions

(15) $$x(0) = a(0), \ x^{(j)}(0) = \sum_{k=0}^{j-1} b_{j-k-1} x^{(k)}(0) + a^{(j)}(0),$$
$$j = 1, 2, \ldots, m-1 \ .$$

The two examples in 8 show that the solution of

$$x(t) = a(t) + \int_0^t (\beta_1 e^{\lambda_1 (t-s)} + \beta_2 e^{\lambda_2 (t-s)}) x(s)\, ds$$

may be found by solving the initial value problem

$$x^{(2)} - (\lambda_1 + \lambda_2 + \beta_1 + \beta_2) x^{(1)} + (\lambda_1 \lambda_2 + \beta_1 \lambda_2 + \beta_2 \lambda_1) x =$$

$$= \lambda_1 \lambda_2\, a(t) - (\lambda_1 + \lambda_2) a^{(1)}(t) + a^{(2)}(t) ,$$

$$x(0) = a(0),\ x^{(1)}(0) = (\beta_1 + \beta_2) a(0) + a^{(1)}(0) .$$

Similarly, the solution of the integral equation

$$x(t) = \sum_{i=0}^{n-1} a_i t^i + \int_0^t (t-s)^{n-1} x(s)\, ds$$

may be obtained by solving the initial value problem

$$x^{(n)} - (n-1)!\, x = 0,\ x^{(j)}(0) = j!\, a_j ,\ j = 0, 1, \ldots, n-1 .$$

Since the kernels of the type $k(t) = \sum_{i=1}^{n} \beta_i e^{\lambda_i t}$ occur quite frequently in many practical problems of the control theory, an explicit formula for the corresponding initial value problem is given below.

10. <u>Example</u>. The solution of the integral equation

$$x(t) = a(t) + \int_0^t \left(\sum_{i=1}^{n} \beta_i e^{\lambda_i (t-s)}\right) x(s)\, ds$$

may be found by solving the initial value problem (14), (15). The numbers A_k in (14) are now the coefficients of the polynomial

$$P(\lambda) = \sum_{k=0}^{n} A_k \lambda^k = \prod_{i=1}^{n} (\lambda - \lambda_i) .$$

It is known that the coefficients A_k may be expressed by the roots λ_i as follows:

$$A_n = 1 ,$$

$$A_{n-k} = (-1)^k \sum_{\substack{i_1, i_2, \ldots, i_k \\ i_1 < i_2 < \ldots < i_k}} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k} ,\ k = 1, 2, \ldots, n .$$

The numbers b_k, $a_k = a^{(k)}(0)$ in (15) are given by the relation

$$b_k = \sum_{i=1}^{m} \beta_i \lambda_i^k , \qquad k = 0, 1, \dots, m-1 .$$

11. Remark. In the analysis of a linear integral equation

(16) $$x(t) = a(t) + \int_0^t b(t-s)\, x(s)\, ds$$

a very important role is played by its resolvent kernel $\varkappa$ given as a solution of a linear integral equation

(17) $$\varkappa(t) = b(t) + \int_0^t b(t-s)\, \varkappa(s)\, ds .$$

It is well-known that if a function $\varkappa$ is a resolvent kernel of Equation (16), then the solution x of (16) may be expressed as

(18) $$x(t) = a(t) + \int_0^t \varkappa(t-s)\, a(s)\, ds .$$

Since the resolvent equation (17) is again a linear Volterra integral equation, Theorem 6 or its modifications in Remarks 7 and 9 may be applied. Thus the following theorem may be formulated:

12. Theorem. Let the function $b \in \mathcal{C}^{(m)}$ be a solution of the equation $\sum_{k=0}^{m} A_k x^{(k)} = 0$. Then the resolvent kernel $\varkappa$ of the equation (16) satisfies the initial value problem

(19) $$\sum_{k=0}^{m} C_k x^{(k)} = 0 , \quad C_k = A_k - \sum_{j=k+1}^{m} A_j b_{j-k-1} ,$$
$$j = 0, 1, \dots, m-1 , \quad C_m = A_m ,$$

with the initial conditions

$$(20)\qquad x^{(j)}(0) = \sum_{k=0}^{j} b_{j-k}\, r^{(k-1)}(0),\ j = 0,1,\dots,m-1,$$

where $r^{(-1)}(0) = 1$.

13. Remark. It may be seen that finding the solution of the initial value problem (19),(20) for the resolvent kernel is easier than solving the initial value problem for the solution of Equation (16) itself. Moreover, if the resolvent kernel r of Equation (16) is known, any solution of Equation (16) with an arbitrary right hand side $a(t)$ is found by integration using Relation (18). On the other hand, when using Equation (14), the corresponding particular integral of this equation has to be computed for each particular choice of the function $a(t)$.

Now, let us apply Theorem 12 to find the resolvent kernel of the integral equation from Example 10.

14. Examples. 1. The resolvent kernel of the linear Volterra integral equation

$$(21)\qquad x(t) = a(t) + \int_0^t \Big(\sum_{i=1}^{m} \beta_i\, e^{\lambda_i (t-s)}\Big)\, x(s)\, ds$$

satisfies the initial value problem (19),(20) with the coefficients A_k, b_k described in Example 10. In a special case, e.g. for $m = 2$, the initial value problem has the form

$$(22)\qquad x^{(2)} - (\lambda_1 + \lambda_2 + \beta_1 + \beta_2)\, x^{(1)} + (\lambda_1 \lambda_2 + \beta_1 \lambda_2 + \beta_2 \lambda_1)\, x = 0,$$

$$x(0) = \beta_1 + \beta_2,\ x^{(1)}(0) = (\beta_1 + \beta_2)^2 + \beta_1 \lambda_1 + \beta_2 \lambda_2$$

Let μ_1, μ_2 be characteristic roots of Equation (22), $\mu_1 \neq \mu_2$. The resolvent kernel r is then

$$r(t) = K_1 e^{\mu_1 t} + K_2 e^{\mu_2 t}$$

with

$$K_1 = \frac{1}{\mu_1 - \mu_2} [(\beta_1 + \beta_2)^2 + \beta_1 \lambda_1 + \beta_2 \lambda_2 - (\beta_1 + \beta_2)\mu_2] ,$$

$$K_2 = \frac{1}{\mu_1 - \mu_2} [(\beta_1 + \beta_2)\mu_1 - (\beta_1 + \beta_2)^2 - \beta_1 \lambda_1 - \beta_2 \lambda_2] .$$

In the special case $b(t) = 1 - e^{-t}$ which often occurs e.g. in the theory of phase controlled oscillations, the resolvent kernel is obtained as the solution of the initial value problem

$$x^{(2)} + x^{(1)} - x = 0, \quad x(0) = 0, \quad x^{(1)}(0) = 1 .$$

Setting $\mu_1 = \frac{-1 + \sqrt{5}}{2}$, $\mu_2 = \frac{-1 - \sqrt{5}}{2}$ the resolvent kernel r of the integral equation with the kernel $b(t) = 1 - e^{-t}$ has the form

$$r(t) = \frac{1}{\sqrt{5}} (e^{\mu_1 t} - e^{\mu_2 t}) .$$

2. The resolvent kernel r for an integral equation with the kernel $b(t) = \sum_{i=0}^{m} a_i t^i$ may be found as the solution of the initial value problem (19),(20) as follows. The polynomial b is the solution of the differential equation $x^{(m+1)} = 0$. Thus $A_{m+1} = 1$, $A_k = 0$ for $k = 0, 1, \ldots, m$, $b_k = k!\, a_k$ for $k = 0, 1, \ldots, m$.

Hence, for the coefficients C_k of Equation (19) we obtain

$$C_0 = 1, \quad C_k = A_{m-k+1} - \sum_{j=1}^{k} A_{m+1-k+j}\, b_{j-1} = -b_{k-1} = -(k-1)!\, a_{k-1}$$

for $k = 1, 2, \ldots, m+1$.

Thus the resolvent kernel solves the initial value problem

$$x^{(n+1)} - a_0 x^{(n)} - a_1 x^{(n-1)} - \dots - (n-1)!\, a_{n-1} x^{(1)} - n!\, a_n x = 0 \ ,$$

$$x^{(j)}(0) = \sum_{k=0}^{j} (j-k)!\, a_{j-k}\, x^{(k-1)}(0), \quad j = 0, 1, 2, \dots, n \ .$$

3. A procedure similar to that described above leads to differential equations for resolvent kernels of integral equations having kernels of the type $b(t) = P(t)\, e^{\lambda t}$ with $P(t)$ a polynomial of the degree $n-1$. It is obvious that the function b is a solution of an ordinary linear differential equation of the order n with constant coefficients

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \lambda^k x^{(n-k)} = 0 \ ,$$

with λ the characteristic root of multiplicity n. Hence

$$A_{n-k} = (-1)^k \binom{n}{k} \lambda^k, \quad b_k = b^{(k)}(0) = \sum_{j=0}^{k} \binom{k}{j} P^{(k-j)}(0)\, \lambda^j \ ,$$

$$C_i = (-1)^i \binom{n}{i} \lambda^i - \sum_{k=1}^{i} (-1)^{i-k} \binom{n}{i-k} \lambda^{i-k} \sum_{j=0}^{k-1} \binom{k-1}{j} P^{(k-j-1)}(0)\, \lambda^j \ .$$

Now, substituting these values of the constants C_k, b_k into (19) and (20), the initial value problem for the resolvent kernel is obtained.

15. Remark. The method described above leads to algebraic equations, whose roots will eventually have to be computed. Here we meet with the same difficulty as when using the Laplace transforms. Notwithstanding, in several special cases our method is more convenient and the pro-

cess of computing the resolvent kernels is very simple. The procedure just described may be modified in many cases in a variety of ways depending on the special form of the kernel. One of such modifications will be shown in what follows.

16. Example. Take the same kernel as in Example 14.3, that is, $k(t) = P(t)e^{\lambda t}$ and the resolvent equation in the form

$$(22) \quad \kappa(t) = P(t)e^{\lambda t} + \varepsilon\int_0^t \kappa(t-s)P(s)e^{\lambda s}\,ds$$

where ε may be either 1 or -1. Setting $v(t) = k(t) = \varepsilon\kappa(t)$, $u(t) = a(t) = P(t)e^{\lambda t}$ into (5.k) we obtain

$$\kappa^{(k)}(t) = \varepsilon\sum_{j=0}^{k-1}\kappa^{(k-j-1)}(0)(P(t)e^{\lambda t})^{(j)} + (P(t)e^{\lambda t})^{(k)} + \varepsilon\int_0^t \kappa^{(k)}(t-s)P(s)e^{\lambda s}\,ds.$$

Denoting $\kappa^{(-1)}(0) = \varepsilon$ and introducing the abbreviated notation $\kappa^{(j)}(0) = \kappa_j$ for $j = -1, 0, 1, 2, \dots$;

$$H_k(t) = \varepsilon\sum_{j=0}^{k}\kappa^{(k-j-1)}(0)(P(t)e^{\lambda t})^{(j)} ,$$

we have

$$\begin{aligned} H_k(t) &= \varepsilon\sum_{j=0}^{k}\kappa_{k-j-1}\sum_{q=0}^{j}\binom{j}{q}P^{(q)}(t)\lambda^{j-q}e^{\lambda t} = \\ &= \varepsilon\sum_{q=0}^{k}P^{(q)}(t)\sum_{j=q}^{k}\kappa_{k-j-1}\binom{j}{q}\lambda^{j-q}e^{\lambda t} = \\ &= \varepsilon\sum_{q=0}^{k}\frac{P^{(q)}(t)}{q!}\sum_{j=q}^{k}j(j-1)\dots(j-q+1)\kappa_{k-j-1}\lambda^{j-q}e^{\lambda t} = \\ &= \varepsilon\sum_{q=0}^{k}\frac{P^{(q)}(t)}{q!}Q_k^{(q)}(\lambda)e^{\lambda t} , \end{aligned}$$

where

(23) $Q_k(\lambda) = Q_k^{(0)}(\lambda) = \sum_{j=0}^{k} \kappa_{k-j-1}\lambda^j$, $Q_k^{(q)}(\lambda) = \sum_{j=q}^{k} j(j-1)\cdot$
$\cdot\ldots\cdot(j-q+1)\,\kappa_{k-j-1}\,\lambda^{j-q}$.

Clearly, the polynomial $Q_k^{(q)}(\lambda)$ is the q-th derivative of the polynomial $Q_k(\lambda)$. Thus the k-th derivative of both the sides of the resolvent equation (22) may be written in the form

$$(24.k)\qquad \kappa^{(k)}(t) = \varepsilon \sum_{q=0}^{k} \frac{P^{(q)}(t)\,Q_k^{(q)}(\lambda)}{q!}\, e^{\lambda t} + \\ + \varepsilon \int_0^t \kappa^{(k)}(t-s)\,P(s)\,e^{\lambda s}\,ds \,.$$

17. Remark. Equation (24.k) appears to be a very useful tool when investigating the various qualitative and quantitative properties of derivatives of the resolvent kernels of Volterra integral equations having kernels of the form $P(t)\,e^{\lambda t}$. One illustration of such applications is given in the next example.

18. Example. Let us investigate the following problem. Does there exist a polynomial $P(t) = a_0 + a_1 t + \ldots$ $\ldots + a_{k-1}\,t^{k-1}$ of the degree $k-1$ such that the resolvent kernel κ , corresponding to the Volterra kernel $k(t) = P(t)\,e^{\lambda t}$, will also be a polynomial of the same degree? We shall find the conditions of the existence of such polynomial. Since κ is required to be a polynomial of the degree $k-1$, its k-th derivative has to be identically zero. Since the function $\kappa^{(k)}$ is a solution of Equation (24.k), the following must hold:

$$\varepsilon \sum_{q=0}^{k} \frac{P^{(q)}(t)\,Q_k^{(q)}(\lambda)}{q!}\, e^{\lambda t} \equiv 0 \,.$$

Hence all coefficients $Q_k^{(q)}(\lambda)$ for $q = 0, 1, \dots, k-1$ have to be zero. Thus λ is the root of multiplicity k of the polynomial $Q_k(\lambda)$. Consequently, $Q_k(x) = \varepsilon.(x-\lambda)^k$. Comparing this with (23) we obtain

$$Q_k(x) = \varepsilon(x-\lambda)^k = \varepsilon \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} x^j \lambda^{k-j} = \sum_{j=0}^{k} \kappa_{k-j-1} x^j ,$$

so that

$$\kappa_{k-j-1} = \varepsilon (-1)^{k-j} \binom{k}{j} \lambda^{k-j}, \quad j = 0, 1, \dots, k .$$

Setting $i = k - j - 1$ we have

$$\kappa_i = (-1)^{i+1} \varepsilon \binom{k}{i+1} \lambda^{i+1} .$$

Since the number $\frac{\kappa_i}{i!}$ is the i-th coefficient of the resolvent kernel κ, we have finally obtained an explicit formula for the resolvent kernel

$$\kappa(t) = \sum_{i=0}^{k-1} (-1)^{i+1} \varepsilon \binom{k}{i+1} \frac{\lambda^{i+1}}{i!} t^i .$$

Now, the coefficients $a_i = \frac{P^{(i)}(0)}{i!}$ of the polynomial $P(t)$ remain to be found. Equation (22) may be rewritten in the form

$$P(t) = \kappa(t) e^{-\lambda t} - \varepsilon \int_0^t \kappa(t-s) e^{-\lambda(t-s)} P(s)\, ds .$$

According to (5.k), the k-th derivative of $P(t)$ is

$$P^{(k)}(t) = -\varepsilon \sum_{j=0}^{k-1} P^{(k-j-1)}(0) (\kappa(t) e^{-\lambda t})^{(j)} + (\kappa(t) e^{-\lambda t})^{(k)} -$$
$$- \varepsilon \int_0^t \kappa(t-s) e^{-\lambda(t-s)} P^{(k)}(s)\, ds .$$

Setting

P_{k-j-1} for $P^{(k-j-1)}(0)$, $j = 0, 1, \dots, k-1$, $P^{(-1)}(0) = P_{-1} = -\varepsilon$,

we obtain

$$\tilde{H}_k(t,\lambda) = -\varepsilon \sum_{j=0}^{k} P_{k-j-1}(r(t)e^{-\lambda t})^{(j)} =$$

$$= -\varepsilon \sum_{j=0}^{k} P_{k-j-1} \sum_{q=0}^{j} \binom{j}{q} \quad r^{(q)}(t)(-\lambda)^{j-q} e^{-\lambda t} =$$

$$= -\varepsilon \sum_{q=0}^{k} \frac{r^{(q)}(t)}{q!} \sum_{j=q}^{k} j(j-1)\cdot \ldots \cdot (j-q+1) P_{k-j-1}(-\lambda)^{j-q} e^{-\lambda t} =$$

$$= -\varepsilon \sum_{q=0}^{k} \frac{r^{(q)}(t)}{q!} \tilde{Q}_k^{(q)}(-\lambda) e^{-\lambda t} \quad ,$$

where

$$\tilde{Q}_k(-\lambda) = \tilde{Q}_k^{(0)}(-\lambda) = \sum_{j=0}^{k} (-1)^j P_{k-j-1} \lambda^j \quad , \tag{25}$$

and

$$\tilde{Q}_k^{(q)}(-x) = \sum_{j=q}^{k} j(j-1)\cdot \ldots \cdot (j-q+1) P_{k-j-1}(-\lambda)^{j-q} \; ,$$

which is the q-th derivative of $\tilde{Q}_k(-\lambda)$.

Since $P(t)$ is a polynomial of the degree $k-1$, necessarily $P^{(k)}(t) = 0$ for all t , and hence, similarly as above, $\tilde{Q}_k^{(q)}(-\lambda) = 0$ for all $q = 0, 1, \ldots$ $\ldots, k-1$. Thus $-\lambda$ is the root of multiplicity k of the polynomial $\tilde{Q}_k(-\lambda)$.

Hence and from (25) we have

$$\tilde{Q}_k^{(x)} = P_{-1}(x+\lambda)^k =$$

$$= \sum_{j=0}^{k} (-\varepsilon) \binom{k}{j} \lambda^{k-j} x^j = \sum_{j=0}^{k} P_{k-j-1} x^j \quad ,$$

and thus

$$P_{k-j-1} = -\varepsilon \binom{k}{j} \lambda^{k-j} \quad .$$

Setting $i = k - j - 1$, we obtain

$$a_i = \frac{P_i}{i!} = -\varepsilon \binom{k}{i+1} \frac{\lambda^{i+1}}{i!} \quad .$$

In this way, the following result has been obtained: for each kernel of the form $b(t) = P(t)e^{\lambda t}$ with $P(t) = -\varepsilon \sum_{i=0}^{k-1} \binom{k}{i+1} \frac{\lambda^{i+1}}{i!} t^i$ the corresponding resolvent kernel κ is the polynomial

$$\kappa(t) = -\varepsilon \sum_{i=0}^{k-1} (-1)^i \binom{k}{i+1} \frac{\lambda^{i+1}}{i!} t^i \quad .$$

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