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REMARKS ON FLOWS IN NETWORK WITH SHORT PATHS

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In [1] Ford and Fulkerson solve the problem of the maximum value of a flow in a network. In the present note we discuss the case that the length of paths is limited. We show that the natural generalization of the main result of Ford and Fulkerson (min-cut max-flows theorem) doe not hold. We give then some estimations on the values of flows with short paths and we show some extremal cases.

Definition. A network is $S = \langle X, R, k, z, s \rangle$ where X is a finite set (the set of vertices of S), $R \subset X \times X$ (the set of edges of S), $k: R \to N$, N is the set of naturals (the capacity function of S), and $z, s \in X$, $z \neq s$ (the source and the sink of S, respectively).

Subnetwork of S is a network $S' = \langle X', R', k', z, s \rangle$ with $X' \subset X$, $R' \subset R$, $k'(r) \leqq k(r)$ $\forall r \in R'$.

Path in S is $\varphi = \langle x_0, x_1, \dots, x_m \rangle$ where x_i are vertices of S, $\langle x_i, x_{i+1} \rangle$ are edges of S, $x_0 = z$, $x_m = s$. Denote $\varphi^\ell = \{ \langle x_i, x_{i+1} \rangle \}_{i=0}^{m-1}$.

m-path in S is a path $\langle x_0, x_1, \dots, x_m \rangle$ with

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$m \leq n$.

n-flow (flow) in S is a subnetwork $T = \langle X_1, R_1, k_1, z, s \rangle$ of S such that there exists a collection $\{c_i\}_{i \in I}$ of n-paths (paths) in S with $k_1(r) = |\{i \in I ; r \in c_i^\ell\}|$. (The paths need not be disjoint.)

Value of an n-flow (flow) T in S is $|I|$. Denote $f_n(S)$ ($f(S)$) the maximum value of an n-flow (flow) in S.

n-cut (cut) in S is $C \subset R$ such that $\varphi^\ell \cap C \neq \emptyset$ for every n-path (path) φ in S.

Value of n-cut (cut) C is $\sum_{r \in C} k(r)$. Denote $c_n(S)$ ($c(S)$) the minimum value of an n-cut (cut) in S. Denote $d_n(S)$ the maximal value of a flow in S, $D = \langle \tilde{X}, \tilde{R}, \tilde{k}, z, s \rangle$ such that

a) for every $r \in \tilde{R}$ there exists an n-flow in S $\langle X_r, R_r, k_r, z, s \rangle$ with $k_r(r) = \tilde{k}(r)$;

b) there exists a collection $\{d_i\}_{i \in I}$ of paths in S which are not n-paths such that for every $r \in \tilde{R}$ $\tilde{k}(r) = |\{i \in I ; r \in d_i^\ell\}|$.

Remark. In [1] flow in S is defined as a subnetwork $T = \langle X_1, R_1, k_1, z, s \rangle$ of S such that for every $x \in X_1$ $z \neq x \neq s$

$$\sum_{\langle x, \xi \rangle \in R_1} k_1(\langle x, \xi \rangle) = \sum_{\langle \eta, x \rangle \in R_1} k_1(\langle \eta, x \rangle) .$$

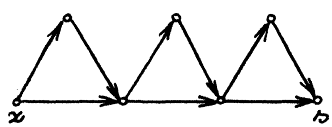
Evidently this definition coincides with ours. The value of T as defined in [1] is $\sum_{\langle z, y \rangle \in R_1} k_1(\langle z, y \rangle)$, which is again equal to the value defined above. In [1] cut is a set $A \subset X$ such that $x \in A$ & $b \in X - A$, value of the cut A is $\sum_{x \in A, y \in X-A, \langle x,y \rangle \in R} k(\langle x, y \rangle)$. It may be easily verified that the minimum value of a cut in S in this sense is just $c(S)$.

Proposition. ("min-cut max-flow theorem".)

$$f(S) = c(S)$$

for every network S.

Remark. The natural generalization of "min-cut max-flow theorem", namely $f_m(S) = c_m(S)$ for every m and every S does not hold - e.g.



$k \equiv 1$: $f_4(S) = 1$, $c_4(S) = 2$.

Theorem. $c_m(S) \geq f_m(S) \geq c_m(S) - d_m(S)$ for every network S and every natural m.

Proof. Let $S = \langle X, R, k, x, b \rangle$ be an arbitrary network, $m \in N$;

A) $c_m(S) \geq f_m(S)$.

Let T be an m-flow in S with the value $f_m(S)$. Then $f_m(S) = f(T) = c(T) \leq c_m(S)$.

B) $f_m(S) \geq c_m(S) - d_m(S)$.

1) $d_m(S) = 0$. Let $E \subset R$ be the set of all edges which are edges of no m-path in S, let $S' = \langle X, R - E, k/R - E, x, b \rangle$. Evidently every path in S' is an m-path and so $f_m(S) \geq f(S')$. Also

$c_m(S) \leq c(S')$ and so

$f_m(S) \geq f(S') = c(S') \geq c_m(S)$, $f_m(S) = c_m(S)$.

2) $d_m(S) > 0$. Let $D = \langle X_1, R_1, k_1, z, s \rangle$ be a flow in S fulfilling the conditions a) b) in the definition of d_m and let the value of D be $d_m(S)$. Denote $S - D = \langle X, \tilde{R}, \tilde{k}, z, s \rangle$, where

$\tilde{R} = (R - R_1) \cup \{ r \in R_1 ; k(r) > k_1(r) \}$,

$\tilde{k} / R - R_1 \equiv k / R - R_1$, $k(r) > k_1(r) \Rightarrow \tilde{k}(r) = k(r) - k_1(r)$.

Evidently $d_m(S - D) = 0$ and so $f_m(S - D) = c_m(S - D)$, further $f_m(S) \geq f_m(S - D)$, $c_m(S - D) + c(D) \geq c_m(S)$ and so

$$f_m(S) \geq f_m(S - D) = c_m(S - D) + c(D) - c(D) \geq$$

$$\geq c_m(S) - c(D) = c_m(S) - f(D) = c_m(S) - d_m(S). \text{ Q.E.D.}$$

<u>Lemma</u>. Let every edge of a network S be an edge of an m-path in S. Then either every path in S is an m-path or there exists an $(m-1)$-path in S.

<u>Proof</u>. Let there be no $(m-1)$-path in S and let $\langle x_0, x_1, \dots, x_k \rangle$ be a path in S with $k > m$. Let $r = \max \{ i ;$ there exists an m-path φ in S with $\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \dots, \langle x_{i-1}, x_i \rangle \in \varphi^{\ell} \}$. Let $\sigma = \langle x_0, x_1, \dots, x_r, y_{r+1}, \dots, y_m \rangle$ be a path in S. According to the assumptions there exists an m-path τ in S with $\langle x_r, x_{r+1} \rangle = \tau^{\ell}$, let $\tau = \langle v_0, v_1, \dots, v_{j-1}, x_r, x_{r+1}, v_{j+2}, \dots, v_m \rangle$. Now if $j \geq r$ then $\langle x_0, x_1, \dots, x_r, x_{r+1}, v_{j+2}, \dots, v_m \rangle$ is an m-path which is in contradiction with the choice of r. If $j < r$, $\langle v_0, v_1, \dots, v_{j-1}, x_r, y_{r+1}, \dots, y_m \rangle$ is an $(m-1)$-path in S, which is a contradiction, too. Q.E.D.

<u>Corollary</u>. $c_m(S) \geqq f_m(S) \geqq c_m(S) - c_{m-1}(S)$, especially $c_{m-1}(S) = 0 \Longrightarrow f_m(S) = c_m(S)$ for every network S and every $m \in N$.

<u>Proof</u>. The special part is an easy consequence of the preceding lemma and theorem. Let $S = \langle X, R, k, x, s \rangle$ let $c_{m-1}(S) > 0$. Let $C \subset R$ be an $(m-1)$-cut in S with $\sum_{\kappa \in C} k(\kappa) = c_{m-1}(S)$. Denote $S^* = \langle X, R - C, k/R - C, x, s \rangle$. Evidently $c_{m-1}(S^*) = 0$ and $c_m(S^*) + c_{m-1}(S) \geqq c_m(S)$ and so

$$f_m(S) \geqq f_m(S^*) = c_m(S^*) \geqq c_m(S) - c_{m-1}(S) \quad . \qquad \text{Q.E.D.}$$

<u>Remark</u>. $f_m(S) = c_m(S) \qquad m = 1, 2, 3$

for every network S.

<u>Remark</u>. It follows easily from the corollary that

$$c_m(S) = f_m(S) \Longrightarrow c_{m+k}(S) \leqq (k+1) . f_{m+k}(S) ;$$

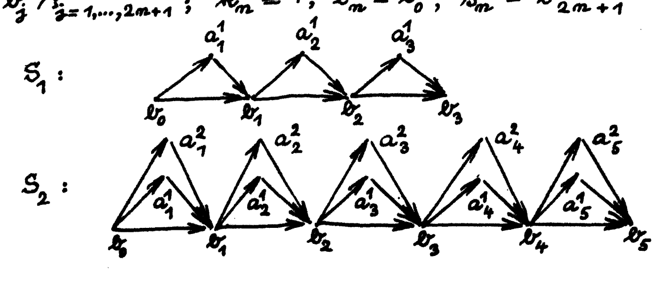
especially $c_m(S) \leqq (m-2) . f_m(S)$.

<u>Remark</u>. If f_m is restricted, c_m is also restricted for a given m. The situation is different if m is arbitrary: For every m there exists a network S_m such that $f_{3m+1}(S_m) = 1$ & $c_{3m+1}(S_m) = m + 1$:

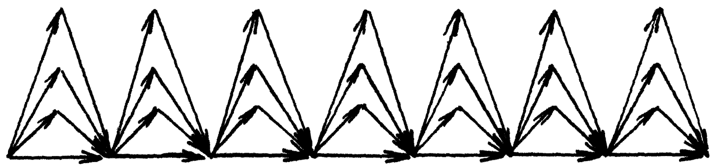
Let $S_m = \langle X_m, R_m, k_m, x_m, s_m \rangle$; $X_m = \{b_j\}_{j=0}^{2m+1} \cup$

$\cup \{a^i_j\}_{j=1,\dots,2m+1}^{i=1,\dots,m}$; $R_m = \{\langle b_{j-1}, b_j \rangle, \langle b_j, a^i_{j+1} \rangle,$

$\langle a^i_j, b_j \rangle\}_{j=1,\dots,2m+1}^{i=1,\dots,m}$; $k_m = 1$; $x_m = b_0$, $s_m = b_{2m+1}$.



S_3 :



Remark. Let $\{q_n\}_{n=1}^{\infty}$ be a series of non-negative integers. There exists a network S such that $f_n(S) = c_n(S) - q_n \ \forall n \in N$ iff $q_1 = q_2 = q_3 = 0$ and there exists k natural with $n > k \Rightarrow q_n = 0$.

Proof. The condition is clearly necessary. Let us prove the sufficiency. Let a network $\mathcal{S}_n$ be given for $n > 3$ with $f_k(\mathcal{S}_n) = c_k(\mathcal{S}_n)$ if $k \neq n$, $f_n(\mathcal{S}_n) = c_n(\mathcal{S}_n) - q_n$.

An example of such a network is e.g.

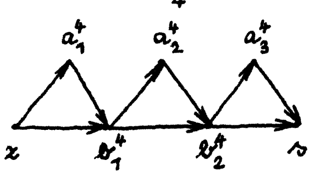
$\mathcal{S}_n = \langle \mathcal{X}_n, \mathcal{R}_n, \mathcal{H}_n, x, s \rangle$. $\mathcal{X}_n = \{a_i^n\}_{i=1}^{n-1} \cup \{b_i^n\}_{i=1}^{n-2} \cup$

$\cup \{x, s\}$;

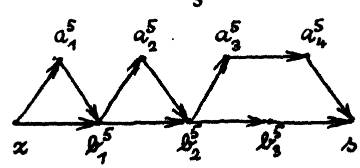
$\mathcal{R}_n = \{\langle x, a_1^n\rangle, \langle x, b_1^n\rangle, \langle a_{n-1}^n, s\rangle, \langle b_{n-2}^n, s\rangle\} \cup \{\langle a_1^n, b_1^n\rangle,$
$\langle b_1^n, a_1^n\rangle, \langle a_2^n, b_2^n\rangle, \langle b_2^n, a_3^n\rangle\} \cup \{\langle b_i^n, b_{i+1}^n\rangle\}_{i=1}^{n-3} \cup \{\langle a_i^n, a_{i+1}^{n-2}\rangle\}_{i=3}^{n-2}$;

$\mathcal{H}_n = q_n$.

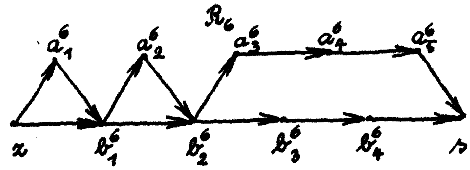
$\mathcal{R}_4$



$\mathcal{R}_5$



$\mathcal{R}_6$



Now, the network we are looking for clearly is $\langle \bigcup_{n=1}^{k} X_n, \bigcup_{n=1}^{k} R_n, \mathcal{H}, x, s \rangle$, where $\mathcal{H} / R_n \equiv \mathcal{H}_n$. Q.E.D.

Remark. We may take into consideration not only the upper bound of the length of paths but also the lower bound. We may define an $m-n$ -path as a path $\langle x_0, x_1, \ldots, x_k \rangle$ with $m \leq k \leq n$ and we may analogously as before define $m-n$ -flow, $m-n$ -cut and $d_{m,n}$. It is easy to see that a little change of the proof of the theorem gives

$$c_{m,n}(S) \geq f_{m,n}(S) \geq c_{m,n}(S) - d_{m,n}(S) .$$

R e f e r e n c e

[1] FORD, FULKERSON: Flows in Networks, New Jersey,1962.

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