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NOTE ON THE FREDHOLM ALTERNATIVE FOR NONLINEAR OPERATORS

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1. Introduction. This note deals with the solving of nonlinear operators' equations $\lambda T x - S x = f$ in dependence on the real parameter λ, where T and S are nonlinear operators defined on a real Banach space X with values in a real Banach space Y. Similar results in linear functional analysis are well-known and they are sometimes called Fredholm theorems. We shall suppose that S is a completely continuous operator and T works as "the identity operator".

This problem was studied in [8],[5],[4] and [1, 1 a]. S.I. Pochožajev supposed in [8] that $Y = X^*$ (X^* is the dual space), T and S are the odd and a-homogeneous operators and X has a Schauder basis.

J. Nečas [5] proved an analogous theorem for the operators T and S which are "near to homogeneous" and $Y = X^*$, X is a complex Banach space. A similar result was proved by M. Kučera in [4] for a real Banach space. The conditions on "near to homogeneity" are stronger than the analogous ones in [5].

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In [1,1 a] , there is established "Fredholm alternative" for the odd operators which are "near to homogeneous" in the sense of [5] whose domain is a real Banach space X and the range is a subset of a real Banach space Y. Both X and Y are supposed to be separable Banach spaces with some special properties, for instance the Banach spaces with the Schauder bases. After the manuscript of this note was sent for printing, the author learned that the same result for the odd and homogeneous operators (and Banach spaces are supposed with the same special properties) had independently been obtained by W.V. Petryshyn [7].

In Section 2, a generalization of results from [1,1 a] is given. The spaces X and Y are not supposed to be separable. The case when T and S are homogeneous operators with different degrees is solved in this section, too.

The main theorems of this note are applied both to the boundary value problem for partial differential equations and the integral equations in Section 3.

2. Main theorems

Unless otherwise stated, we shall suppose that X and Y are real Banach spaces with the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. We shall use the symbols " $\longrightarrow$ " , " $\rightharpoonup$ " to denote the strong and weak convergences, respectively.

Definition 1. Let T be a mapping defined on X with values in Y ($T : X \longrightarrow Y$). T is said to be a (K, L, a)-homeomorphism of X onto Y if

(1) T is a homeomorphism of X onto Y,

(2) there exist real numbers $K > 0$, $a > 0$, $L \geq 0$ such that

$$L \|x\|_X^a \leq \|Tx\|_Y \leq K \|x\|_X^a$$

for each $x \in X$.

<u>Lemma</u>. Let $T: X \longrightarrow Y$ be a (K, L, a)-homeomorphism of X onto Y with $L = 0$, $S: X \longrightarrow Y$ and $\lambda \neq 0$ a real number.

a) If $\lim\limits_{\|x\|_X \to \infty} \|\lambda Tx - Sx\|_Y = \infty$, then

$$\lim_{\|y\|_Y \to \infty} \|y - ST^{-1}(\frac{y}{\lambda})\|_Y = \infty .$$

b) If $y - ST^{-1}(\frac{y}{\lambda})$ is a mapping of Y onto Y, then $\lambda T - S$ is a mapping of X onto Y.

<u>Proof</u>. a) Suppose that there exist a sequence $\{y_m\}$, $y_m \in Y$, $\|y_m\|_Y \to \infty$ and a real number $A > 0$ such that

$$\|y_m - ST^{-1}(\frac{y_m}{\lambda})\|_Y \leq A$$

for each positive integer m.

It is obvious that there exists $x_m \in X$ such that $y_m = \lambda T x_m$. The inequality

$$\|y_m\|_Y = |\lambda| \, \|Tx_m\|_Y \leq K |\lambda| \, \|x_m\|_X^a$$

gives $\|x_m\|_X \longrightarrow \infty$ and $\|\lambda Tx_m - Sx_m\|_Y \leq A$, a contradiction with $\|\lambda Tx_m - Sx_m\|_Y \longrightarrow \infty$.

b) Let $z_0 \in Y$. According to the assumptions there exist $y_0 \in Y$ such that $y_0 - ST^{-1}(\frac{y_0}{\lambda}) = z_0$ and $x_0 \in X$ such that $\lambda T x_0 = y_0$. Then

$\lambda T x_0 - S x_0 = z_0$ and $\lambda T - S$ is onto Y.

<u>Theorem 1.</u> Let T be an odd (K, L, a)-homeomorphism of X onto Y with $L = 0$ and $S: X \longrightarrow Y$ an odd completely continuous operator (i.e. S is continuous and it transforms every bounded subset of X on a compact subset of Y). Let $\lambda \neq 0$ be a real number. Suppose that

$$\lim_{\|x\|_X \to \infty} \|\lambda T x - S x\|_Y = \infty .$$

Then $\lambda T - S$ maps X onto Y .

<u>Proof</u>. It is obvious that $S T^{-1}: Y \longrightarrow Y$ is an odd completely continuous operator. The lemma implies

$$\lim_{\|y\|_Y \to \infty} \|y - S T^{-1}(\frac{y}{\lambda})\|_Y = \infty .$$

Let $z_0 \in Y$. There exists $R > 0$ such that $\|y - S T^{-1}(\frac{y}{\lambda})\|_Y > \|z_0\|_Y \geq 0$ for each $y \in Y$, $\|y\|_Y = R$.

According to the properties of the Leray-Schauder degree and the Borsuk-Ulam theorem we have that

$d\,[\,y - S T^{-1}(\frac{y}{\lambda}), K_R, \theta_Y\,]$ is an odd number (i.e. different from zero), where $d\,[\,y - S T^{-1}(\frac{y}{\lambda}), K_R, \theta_Y\,]$ is the Leray-Schauder degree of the mapping $y - S T^{-1}(\frac{y}{\lambda})$ on the open ball $K_R = \{ y \in R, \|y\|_Y < R \}$ with respect to the zero point θ_Y . For each $y \in Y$, $\|y\|_Y = R$ and all $t \in \langle 0, 1 \rangle$ there is

$$\|y - S T^{-1}(\frac{y}{\lambda}) - t z_0\|_Y \geq \|y - S T^{-1}(\frac{y}{\lambda})\|_Y - \|z_0\|_Y > 0$$

and thus from the homotopy property of degree we have

$$d\,[y - ST^{-1}(\frac{y}{\lambda}), K_R, x_0] =$$
$$= d\,[y - ST^{-1}(\frac{y}{\lambda}), K_R, \theta_Y] \neq 0 .$$

The previous fact implies the existence of $y_0 \in K_R$ for which $y_0 - ST^{-1}(\frac{y_0}{\lambda}) = x_0$. From this follows that $y - ST^{-1}(\frac{y}{\lambda})$ maps Y onto Y and the lemma proves the theorem. (For the properties of the Leray-Schauder degree used in this proof see [3].)

Theorem 2. Let T be an odd (K, L, a)-homeomorphism of X onto Y with $L > 0$ and $S: X \longrightarrow Y$ an odd completely continuous operator. Suppose that

$$\lim_{\|x\|_X \to \infty} \frac{\|Sx\|_Y}{\|x\|_X^a} = A \in E_1 .$$

Then for $|\lambda| \notin \langle \frac{A}{K}, \frac{A}{L} \rangle \cup \{0\}$ the operator $\lambda T - S$ maps X onto Y.

Proof. For proving the assertion it is sufficient to show that

$$\lim_{\|x\|_X \to \infty} \|\lambda Tx - Sx\|_Y = \infty .$$

Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\}$, $x_n \in X$, $\|x_n\|_X \longrightarrow \infty$ such that $\|\lambda Tx_n - Sx_n\|_Y \leq M$. From this

$$\frac{\lambda Tx_n}{\|x_n\|_X^a} - \frac{Sx_n}{\|x_n\|_X^a} \longrightarrow \theta_Y \text{ and } \frac{|\lambda|\,\|Tx_n\|_Y}{\|x_n\|_X^a} \longrightarrow A .$$

(θ_Y is the zero element of Y.)

But $|\lambda| K \geq A \geq |\lambda| L$, a contradiction with

$|\lambda| \notin \langle \frac{A}{K}, \frac{A}{L} \rangle$.

Corollary 1. Let the assumptions of the preceding theorem hold with $A = 0$.
Then for each $\lambda \neq 0$ the operator $\lambda T - S$ maps X onto Y .

Definition 2. Let X and Y be two Banach spaces, $T: X \to Y$, $a > 0$.

a) T is said to be a-homogeneous if $T(tu) = t^a Tu$ for each $t \geq 0$ and all $u \in X$.

b) T is said to be a-quasihomogeneous with respect to T_0 , if there exists an operator $T_0 : X \to Y$, T_0 is a-homogeneous and if $t_n \searrow 0$, ($t_1 \geq t_2 \geq \ldots \geq t_n \geq t_{n+1} \geq \ldots > 0$ and $t_n \to 0$), $u_n \rightharpoonup u_0$, $t_n^a T(\frac{u_n}{t_n}) \to g \in Y$, then $T_0 u_0 = g$.

c) T is said to be a-strongly quasihomogeneous with respect to T_0 , if there exists an operator $T_0 : X \to Y$, T_0 is a-homogeneous and $t_n \searrow 0$, $u_n \rightharpoonup u_0$ imply $t_n^a T(\frac{u_n}{t_n}) \to T_0 u_0$.

Remark 1. (See [1,1 a] .) If $S: X \to Y$ is a-strongly quasihomogeneous with respect to S_0 , then S_0 is strongly continuous (i.e. $x_n \rightharpoonup x_0$ implies $S_0 x_n \to S_0 x_0$). If $T: X \to Y$ is a-homogeneous, then T is a-quasihomogeneous with respect to T provided that T is strongly closed (i.e. $x_n \rightharpoonup x_0$, $Tx_n \to y$ imply $Tx_0 = y$) and T is a-strongly quasihomogeneous with respect to T provided that T is strongly continuous.

Corollary 2. Let T be an odd (X, L, a)-homeomorphism of X onto Y with $L > 0$, $S: X \to Y$ an odd completely continuous b-strongly quasihomogeneous operator with respect to S_0, $a > b$, $\lambda \neq 0$ and X a reflexive Banach space.

Then $\lambda T - S$ maps X onto Y.

Proof. According to Corollary 1 it is sufficient to prove that

$$\lim_{\|x\|_X \to \infty} \frac{\|Sx\|_Y}{\|x\|_X^a} = 0 .$$

Suppose that there exist $\varepsilon > 0$ and a sequence $\{x_n\}$, $x_n \in X$, $\|x_n\|_X \to \infty$ such that

$$\frac{x_n}{\|x_n\|_X} = v_n \longrightarrow v_0 \quad \text{and} \quad \frac{\|Sx_n\|_Y}{\|x_n\|_X^a} \geq \varepsilon$$

for each positive integer n.

Then

$$\frac{S(\|x_n\|_X v_n)}{\|x_n\|_X^b} \longrightarrow S_0 v_0 \quad ,$$

and because $\dfrac{\|x_n\|_X}{\|x_n\|_X^a} \longrightarrow 0$ we have

$$0 < \varepsilon \leq \frac{\|Sx_n\|_Y}{\|x_n\|_X^a} = \frac{\|x_n\|_X^b}{\|x_n\|_X^a} \cdot \frac{\|Sx_n\|_Y}{\|x_n\|_X^b} \longrightarrow 0 .$$

It is a contradiction.

Definition 3. Let X and Y be two Banach spaces, $a > 0$, $T_0 : X \longrightarrow Y$, $S_0 : X \longrightarrow Y$ a-homogeneous operators and $\lambda \neq 0$ a real number.

λ is said to be an eigenvalue for the couple (T_0, S_0) if there exists $u_0 \in X$, $u_0 \neq \theta_X$ (θ_X is the zero element of X) such that $\lambda T_0 u_0 - S_0 u_0 = \theta_Y$.

<u>Theorem 3.</u> Let T be an odd (X, L, a)-homeomorphism of X onto Y with $L > 0$ and a-quasihomogeneous operator with respect to T_0. Let $S : X \to Y$ be an odd completely continuous a-strongly quasihomogeneous operator with respect to S_0.

If $\lambda \neq 0$ is not an eigenvalue number for the couple (T_0, S_0) and X is a reflexive Banach space, then $\lambda T - S$ maps X onto Y.

<u>Proof</u>. It suffices to show that

$$\lim_{\|x\|_X \to \infty} \|\lambda T x - S x\|_Y = \infty .$$

Suppose that there exist a sequence $\{x_n\}$, $x_n \in X$, $\|x_n\|_X \to \infty$ and a constant $M > 0$ such that

$$\frac{x_n}{\|x_n\|_X} = v_n \longrightarrow v_0 \quad \text{and} \quad \|\lambda T x_n - S x_n\|_Y \le M$$

for each positive integer n.

Then

$$\frac{\lambda T(\|x_n\|_X v_n)}{\|x_n\|_X^a} - \frac{S(\|x_n\|_X v_n)}{\|x_n\|_X^a} \longrightarrow \theta_Y ,$$

$$\frac{S(\|x_n\|_X v_n)}{\|x_n\|_X^a} \longrightarrow S_0 v_0 ,$$

$$\frac{\lambda T(\|x_n\|_X v_n)}{\|x_n\|_X^a} \longrightarrow S_0 v_0 , \quad \text{and}$$

$\lambda T_0 v_0 - S_0 v_0 = \theta_Y$.

Because $\frac{\| \lambda T x_n \|_Y}{\| x_n \|^a} \geq |\lambda| L > 0$ we have $S_0 v_0 \neq \theta_Y$ and $v_0 \neq \theta_X$. Thus λ is the eigenvalue number for the couple (T_0 , S_0) which is a contradiction.

Remark 2. If $\lambda \neq 0$ is an eigenvalue number for the couple (T_0 , S_0) , then the operator $\lambda T - S$ can map X onto Y . Set $X = Y = E_1$ (the real numbers with usual topology) and $Tx = x^3$, $Sx = \frac{|x|}{1+|x|} \cdot x^3$.

Then $T_0 x = x^3$, $S_0 x = x^3$ and $\lambda = 1$ is the eigenvalue number for the couple (T_0 , S_0) . But

$$\lim_{|x| \to \infty} | x^3 - \frac{|x|}{1+|x|} x^3 | = \infty$$

and according to Theorem 1 the operator $T - S$ maps E_1 onto E_1 .

Remark 3. Let X and Y be two finitedimensional Banach spaces. Suppose that T is an odd (X, L, a)-homeomorphism of X onto Y with $L > 0$ and $S : X \to Y$ is an odd continuous and b-strongly quasihomogeneous operator with respect to S_0 . Let $S_0 v = \theta_Y$ imply $v = \theta_X$.

If $a < b$, $\lambda \neq 0$, then $\lambda T - S$ maps X onto Y .

Proof. We shall prove

$$\lim_{\| x \|_X \to \infty} \| \lambda T x - S x \|_Y = \infty .$$

Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\}$, $x_n \in X$, $\|x_n\|_X \to \infty$ such that

$\dfrac{x_n}{\|x_n\|_X} = v_n \to v_0$ and $\|\lambda T x_n - S x_n\|_Y \leq M$

for each positive integer n.

Then

$$\frac{\lambda T(\|x_n\|_X \cdot v_n)}{\|x_n\|_X^b} - \frac{S(\|x_n\|_X \, v_n)}{\|x_n\|_X^b} \to \theta_Y ,$$

$$\frac{\lambda T(\|x_n\|_X \, v_n)}{\|x_n\|_X^b} \to S_0 v_0 .$$

But $K|\lambda| \dfrac{\|x_n\|_X^a}{\|x_n\|_X^b} \geq \dfrac{\|\lambda T x_n\|_Y}{\|x_n\|_X^b} \geq L|\lambda| \dfrac{\|x_n\|_X^a}{\|x_n\|_X^b}$,

$\dfrac{\|\lambda T x_n\|_Y}{\|x_n\|_X^b} \to 0$ and $S_0 v_0 = \theta_Y$.

From our assumption $v_0 = \theta_X$ and this is a contradiction with $\|v_0\|_X = 1$.

3. Applications

Example 1. Let Ω be a bounded domain in E_N with a smooth boundary $\partial\Omega$. By $\overset{\circ}{W}{}_p^{(1)}(\Omega)$ we denote the well-known Sobolev space.

Let $f \in (\overset{\circ}{W}{}_2^{(1)}(\Omega))^*$, $s > 0$, $\lambda \neq 0$. The weak solution of a nonlinear boundary value problem

$$(1) \quad \begin{cases} -\lambda \sum_{i=1}^{N} \dfrac{\partial^2 u}{\partial x_i^2} - u \cdot \dfrac{|u|^s}{1+|u|^s} = f \\ \qquad\qquad\qquad u = 0 \text{ on } \partial\Omega \end{cases}$$

is a function $u \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ such that for any

$v \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ there is

$$\lambda \int_\Omega \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_\Omega u \cdot \frac{|u|^s}{1+|u|^s} \cdot v\, dx = \int_\Omega f \cdot v\, dx .$$

Denoting by (w^*, u) the pairing between $(\overset{\circ}{W}{}_2^{(1)}(\Omega))^*$ and $\overset{\circ}{W}{}_2^{(1)}(\Omega)$ we can define the operators $T, S, S_0 : \overset{\circ}{W}{}_2^{(1)}(\Omega) \to (\overset{\circ}{W}{}_2^{(1)}(\Omega))^*$ putting

$$(Tu, v) = \int_\Omega \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx ,$$

$$(Su, v) = \int_\Omega u \cdot \frac{|u|^s}{1+|u|^s} \cdot v\, dx ,$$

$$(S_0 u, v) = \int_\Omega u \cdot v\, dx .$$

The assumptions of the theorem 3 are satisfied and thus the boundary value problem (1) has a weak solution provided $\frac{1}{\lambda}$ is not an eigenvalue number of the homogeneous Dirichlet problem for the Laplace operator.

Example 2. The boundary value problem

$$(2) \qquad \begin{cases} -\lambda \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} \right)^3 - |u|^{m-1} \cdot u = f \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has for $1 \leq m < 3$ the weak solution $u \in \overset{\circ}{W}{}_4^{(1)}(\Omega)$ for each $f \in (\overset{\circ}{W}{}_4^{(1)}(\Omega))^*$ and $\lambda \neq 0$.

For $m = 3$ the same problem has a weak solution for arbitrary $f \in (\overset{\circ}{W}{}_4^{(1)}(\Omega))^*$ provided

$$(2') \quad \begin{cases} -\lambda \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} \right)^3 - |u|^2 u = 0 \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

has a trivial solution only. (See Corollary 2 and Theorem 3.)

Example 3. The same result as in Example 2 takes place for the problem

$$(3) \quad \begin{cases} -\lambda \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} \right)^3 - (1 + |u|^{m-1}) u = f \\ u = 0 \quad \text{on } \partial\Omega \, . \end{cases}$$

Example 4. Let M be a compact set in E_N ; $p > 1$, $q \geq 1$, $\lambda \neq 0$ real numbers such that $\frac{p}{q} \geq 1$. Let $K(x, y)$ be a continuous function on $M \times M$.

If $m < \frac{p}{q}$ then for each $F \in L_q$ there exists $u \in L_p$ such that

$$(4) \quad \lambda |u|^{\frac{p}{q}-1} u - \int_M K(x, y) |u(y)|^{m-1} u(y) \, dy = F$$

(see Theorem 2).

If $m = \frac{p}{q}$, then the equation (4) has for each $F \in L_q$ the solution $u \in L_p$ provided the homogeneous equation

$$(4') \quad \lambda |u|^{\frac{p}{q}-1} u - \int_M K(x, y) |u(y)|^{\frac{p}{q}-1} u(y) \, dy = 0$$

has the trivial solution only (see Theorem 3).
(To prove the validity of the assumptions in Theorems 2 and 3 we must use the results on the continuity of the Němycki´

operator and the complete continuity of the Hammerstein's operator - see [3] and [9].)

Remark 4. Some other examples of the differential and integral equations from [5],[6] and [1,1 a] can be solved by this way.

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