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Jaroslav Drahoš

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A CONSTRUCTION OF THE PROJECTIVE MODIFICATION FOR A CLOSURE-SET OF A PRESHEAF

J. PECHANEC - DRAHOŠ, Praha

The purpose of this paper is to generalize and to simplify a theorem concerning the modifications of closure collections of presheaves of closure space of [2]. The method used here is different from that of [2], it is easier and better for understanding the problem.

1. Notations. If X is a closure space with a closure t, then every filter-base of t-neighborhoods of a point $x \in X$ is denoted by $\Delta(x, t)$. If a closure t' is finer than t, we write $t' \leq t$. If T is a set of closures on X, let $\underrightarrow{\lim}\, T$, respectively $\underleftarrow{\lim}\, T$ be the finest, respectively the coarsest closure on X, coarser, respectively finer than each closure from T.

If X is a topological space, we denote by $\mathcal{B}(X)$ the set of all its open subsets. Then for $\mathcal{U} \in \mathcal{B}(X)$ let $\Pi(\mathcal{U})$ respectively $\Pi_0(\mathcal{U})$ be the set of all open coverings (respectively of all finite open coverings) of $\mathcal{U}$.

If X is locally compact, let $P(\mathcal{U})$ be the set of all coverings $\mathcal{V} \in \Pi(\mathcal{U})$ with the following property: $V \in \mathcal{V}$ implies $\overline{V} \subset \mathcal{U}$ is compact. Moreover, $\mathcal{V}_0(\mathcal{U}) = \{V;\ V \in \mathcal{B}(\mathcal{U});\ \overline{V} \subset \mathcal{U}$ is compact$\}$. Thus

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$\mathcal{V}_0(\mathcal{U}) \in P(\mathcal{U})$.

<u>2. Definitions</u>. Let $\mathcal{S} = \{(S_\mathcal{U}\ \tau_\mathcal{U});\ \rho_{\mathcal{U}V};\ X\}$ be a presheaf of sets with closures over a topological space X , i.e. for every $\mathcal{U} \in \mathcal{B}(X)$ let $S_\mathcal{U}$ be a set with a closure $\tau_\mathcal{U}$ and for $\mathcal{U}$, $V \in \mathcal{B}(X)$, $V \subset \mathcal{U}$ let $\rho_{\mathcal{U}V} : S_\mathcal{U} \to S_V$ be a map (not necessarily continuous) of $(S_\mathcal{U}\ \tau_\mathcal{U})$ into $(S_V\ \tau_V)$. The set $\mu = \{\tau_\mathcal{U};\ \mathcal{U} \in \mathcal{B}(X)\}$ (briefly $\mu = \{\tau_\mathcal{U}\}$) is called the set of closures of the presheaf $\mathcal{S}$ (briefly the set of closures). If $\mu = \{\tau_\mathcal{U}\}$, $\mu' = \{\tau'_\mathcal{U}\}$ are two such sets and for every $\mathcal{U} \in \mathcal{B}(X)$ we have $\tau_\mathcal{U} \leq \tau'_\mathcal{U}$, we write $\mu \leq \mu'$. If every $\rho_{\mathcal{U}V} : (S_\mathcal{U}\ \tau_\mathcal{U}) \to (S_V\ \tau_V)$ is continuous, we call $\mathcal{S}$ the presheaf of closure spaces and the set of its closures μ is called the collection of closures of $\mathcal{S}$, briefly the collection.

If $\mathcal{U} \in \mathcal{B}(X)$, $\mathcal{V} \in \Pi(\mathcal{U})$, we have a collection of maps

(3) $\quad Z_\mathcal{V} = \{\rho_{\mathcal{U}V};\ \rho_{\mathcal{U}V} : S_\mathcal{U} \to (S_V\ \tau_V);\ V \in \mathcal{V}\}$

of the set $S_\mathcal{U}$ (the closure $\tau_\mathcal{U}$ is not considered now) into the closure spaces $(S_V\ \tau_V)$, $V \in \mathcal{V}$. Using the set $Z_\mathcal{V}$, we can construct projectively a new closure in $S_\mathcal{U}$. We denote

$$(4) \qquad \tau_{\mathcal{U}\mathcal{V}} = \varprojlim_{V \in \mathcal{V}} \tau_V \quad .$$

As in [1], the set $\mu = \{\tau_\mathcal{U}\}$ is called projective (respectively finitely projective), if $\tau_{\mathcal{U}\mathcal{V}} = \tau_\mathcal{U}$ for every $\mathcal{U} \in \mathcal{B}(X)$ and every $\mathcal{V} \in \Pi(\mathcal{U})$ (respectively every $\mathcal{V} \in \Pi_0(\mathcal{U})$) - see [1], Definition 1.1. , 1.1.26. It is obvious that the projective or finitely projective set μ is a closure collection, i.e. every $\rho_{\mathcal{U}V} : (S_\mathcal{U}\ \tau_\mathcal{U}) \to (S_V\ \tau_V)$

is continuous.

In [2] we proved the following statement (Theorem 1.1.6).

<u>Theorem A.</u> For every collection $\mu = \{\tau_{\mathcal{U}}\}$ of $\mathcal{S}$ there exists a collection $\mu' = \{\tau'_{\mathcal{U}}\}$ such that

a) $\mu \leq \mu'$,

b) μ' is projective,

c) if $\nu = \{\tilde{\tau}_{\mathcal{U}}\}$ is a projective collection and $\mu \leq \leq \nu \leq \mu'$, then $\nu = \mu'$.

The collection μ' was called in [2] projective modification of μ .

In [2] we have proved a theorem concerning the projective modifications of closure collections, which states (1.1.37):

<u>Theorem B</u>. If $\mathcal{S}$ in (2) is a presheaf of closure spaces over a locally compact space X and if its set of closures $\mu = \{\tau_{\mathcal{U}}\}$ is finitely projective, then $\mu' = \{\tau_{\mathcal{U}\mathcal{V}_0(\mathcal{U})}\,;\, \mathcal{U} \in \mathcal{B}(X)\}$. In the following we will not assume μ is finitely projective collection, but only that μ is a set of closures of $\mathcal{S}$ with the property:

(5) Assumption: If $\mathcal{U} \in \mathcal{B}(X)$, $a \in \mathcal{S}_{\mathcal{U}}$, $W \in \Delta(a;\, \tau_{\mathcal{U}})$, $\mathcal{V} \in \Pi_0(\mathcal{U})$, then for every $V \in \mathcal{V}$ there exists $W^V \in \Delta\,(\varphi_{\mathcal{U}V}(a);\, \tau_V)$ such that $\bigcap_{\mathcal{V}} \varphi^{-1}_{\mathcal{U}V}(W^V) \subset W$.

Under the assumption (5) we construct for μ certain projective closure collection $\mu^{\#}$ of $\mathcal{S}$ and we will show that $\mu^{\#}$ is a projective modification of μ not only for the collection μ as in [1] (see Theorem B), but moreover for some sets of closures μ of $\mathcal{S}$.

6. <u>Remark</u>. Since the property (5) does not imply the

continuity of ρ_{UV} , the set of closures with this property is not nececcarily a closure collection. However, it can be easily seen that if μ is a collection, then (5) is equivalent with the finite projectivity of μ .

7. Theorem. Let $\mathcal{S} = \{(S_U\, \tau_U); \rho_{UV}; X\}$ be a presheaf of sets S_U with closures τ_U over a locally compact space X .
Let the set $\mu = \{\tau_U\}$ of closures satisfy Condition (5). Then $\mu^\# = \{\tau_U^\#\} = \{\tau_{U\mathcal{V}_0(U)}; U \in \mathcal{B}(X)\}$ is a projective closure collection of $\mathcal{S}$.

Proof. Obviously, $\mu^\#$ is a collection.
Thus we are going to prove it is projective. Let

$U \in \mathcal{B}(X),\ a \in S_U,\ \mathcal{V} \in \Pi(U),\ W \in \Delta(a; \tau_U^\#)$.

We may assume that $W \supset \bigcap_{i=1}^{m} \rho_{UV_i}^{-1}(W_i)$, where $V_i \in \mathcal{V}_0(U),\ W_i \in \Delta(\rho_{UV_i}(a); \tau_{V_i})$, $i = 1, \dots, m$.
The compactness of the set $\overline{V}_i \subset U$ implies there exist $U_i^1, \dots, U_i^{k_i} \in \mathcal{V}$ such that $\bigcup_{j=1}^{k_i} U_i^j \supset \overline{V}_i$, $i = 1, \dots, m$. Let us set $S_i^j = U_i^j \cap V_i$, $\mathcal{V}_i' = \{S_i^1, \dots, S_i^{k_i}\}$. Then $\mathcal{V}_i' \in \Pi_0(V_i)$, $i = 1, \dots, m$. The local compactness of X implies: There exist $R_i^1, \dots, R_i^{k_i} \in \mathcal{B}(V_i)$ such that

1) $R_i^j \subset S_i^j$,

2) $\overline{R}_i^j \subset U_i^j$ is compact,

3) $\{R_i^1, \dots, R_i^{k_i}\} \in \Pi_0(V_i)$, $i = 1, \dots, m,\ j = 1, \dots, k_i$.

Since μ satisfies (5), we have $\bigcap_{j=1}^{k_i} \rho_{V_i R_i^j}^{-1}(W_i^j) \subset W_i$ for some $W_i^j \in \Delta(\rho_{UR_i^j}(a); \tau_{R_i^j})$, $i = 1, \dots, m$, $j = 1, \dots, k_i$. For these i, j let us set $\widetilde{W}_i^j = \rho_{U_i^j R_i^j}^{-1}(W_i^j)$. Then $\widetilde{W}_i^j \in \Delta(\rho_{UU_i^j}(a); \tau_{U_i^j}^\#)$, because $R_i^j \in \mathcal{V}_0(\mathcal{U}_i^j)$. Moreover, $\mathcal{U}_i^j \in \mathcal{V}$ and $\bigcap_{i=1}^{m} \bigcap_{j=1}^{k_i} \rho^{-1}_{\mathcal{U}\mathcal{U}_i^j}(\widetilde{W}_i^j) \subset W$, which proves the projectivity of $\mu^{\#}$.

<u>8. Remark.</u> The inequality $\mu \leq \mu^{\#}$ is not necessarily true if μ is not a <u>collection</u> of closures of $\mathcal{S}$, but only a <u>set of closures of</u> $\mathcal{S}$. It can be easily seen that this inequality is true iff the set $\mu = \{\tau_{\mathcal{U}}\}$ satisfies the following condition: For every $\mathcal{U}, \mathcal{V} \in \mathcal{B}(X)$, $\overline{\mathcal{V}} \subset \mathcal{U}$ compact, the map $\rho_{\mathcal{U}\mathcal{V}}$ is continuous.

Now let $\mathcal{S} = \{(S_{\mathcal{U}}\tau_{\mathcal{U}}); \rho_{\mathcal{U}\mathcal{V}}; X\}$ be a presheaf of sets with closures, $\mu = \{\tau_{\mathcal{U}}\}$ its set of closures. It can be easily seen (or see [2], 1.1.3) that if $\{\nu^{\alpha} = \{\tau^{\alpha}_{\mathcal{U}}\}; \alpha\}$ is some family of closure collections of $\mathcal{S}$, then $\varprojlim_{\alpha} \nu^{\alpha} = \{\varprojlim_{\alpha} \tau^{\alpha}_{\mathcal{U}}; \mathcal{U} \in \mathcal{B}(X)\}$ and $\varinjlim_{\alpha} \nu^{\alpha} = \{\varinjlim_{\alpha} \tau^{\alpha}_{\mathcal{U}}; \mathcal{U} \in \mathcal{B}(X)\}$ are again closure collections of $\mathcal{S}$.

Therefore we have that for the <u>set</u> μ there exists a <u>collection</u> $\mu^{\sigma} = \{\tau^{\sigma}_{\mathcal{U}}\}$ of $\mathcal{S}$ such that $\mu \leq \mu^{\sigma}$ and if ν is any collection of $\mathcal{S}$ such that $\mu \leq \nu$, then $\mu^{\sigma} \leq \nu$.

<u>9. Definition</u>. Let $\mathcal{S}$ be a presheaf, μ its set of closures. A closure collection $\mu^1 = \{\tau^1_{\mathcal{U}}\}$ of $\mathcal{S}$ is called projective modification of μ if

1) μ^1 is a projective collection,
2) $\mu \leq \mu^1$,
3) if ν is a projective collection of $\mathcal{S}$, such that $\mu \leq \nu$, then $\mu^1 \leq \nu$.

The theorem A implies immediately the existence of the projective modification μ^1 to every set of closures μ, since for μ there exists μ^σ which is a collection and for μ^σ we get the projective modification $(\mu^\sigma)'$ by Theorem A. Obviously $(\mu^\sigma)'$ satisfies the conditions of (9).

If μ is a collection, then the modifications μ' (by Theorem A) and μ^1 (by 9) are equal. Thus we will denote in the following the projective modification of the set μ by μ' instead of μ^1.

10. Theorem. Let $\mathcal{S}$ be a presheaf of sets with closures over a locally compact space X described in (2) which satisfies Condition 5 and let $\mu = \{\tau_{\mathcal{U}}\}$ be its set of closures. Suppose for every $\mathcal{U}, \mathcal{V} \in \mathcal{B}(X)$ with the property: $\overline{\mathcal{V}} \subset \mathcal{U}$ compact, the map $\rho_{\mathcal{U}\mathcal{V}}$ is continuous. Then $\mu^\# = \mu'$ (see 7).

Proof. According to Remark 8 it follows from the continuity of $\rho_{\mathcal{U}\mathcal{V}}$ for $\overline{\mathcal{V}} \subset \mathcal{U}$ compact, that $\mu \leq \mu^\#$, where $\mu^\# = \{\tau^\#_{\mathcal{U}}\} = \{\tau_{\mathcal{U}\mathcal{V}_0(\mathcal{U})}\}$ and $\tau_{\mathcal{U}\mathcal{V}_0(\mathcal{U})} = \varinjlim_{\mathcal{V} \in \mathcal{V}_0(\mathcal{U})} \tau_{\mathcal{V}}$ (see 1). Let us set $\mu' = \{\tau'_{\mathcal{U}}\}$. Since $\tau_{\mathcal{U}} \leq \tau'_{\mathcal{U}}$ for all $\mathcal{U} \in \mathcal{B}(X)$ and since μ is projective, we get $\tau^\#_{\mathcal{U}} = \varinjlim_{\mathcal{V} \in \mathcal{V}_0(\mathcal{U})} \tau_{\mathcal{V}} \leq \varinjlim_{\mathcal{V}_0(\mathcal{U})} \tau'_{\mathcal{V}} = \tau'_{\mathcal{U}}$. Thus $\mu^\# \leq \mu'$. By 7, $\mu^\#$ is a projective collection, therefore by Theorem A and the definition 9 $\mu = \mu^\#$.

11. Remark. For $\mathcal{U} \in \mathcal{B}(X)$ we set (as in [1], 1.1.10)

$$\text{(12)} \qquad \tau^*_{\mathcal{U}} = \varinjlim_{\mathcal{V} \in \Pi(\mathcal{U})} \tau_{\mathcal{U}\mathcal{V}}, \quad \mu^* = \{\tau^*_{\mathcal{U}}\} .$$

Obviously $\tau_{\mathcal{U}} \leq \tau^*_{\mathcal{U}} \leq \tau^1_{\mathcal{U}}$. If X is locally compact, every $\mathcal{V} \in \Pi(\mathcal{U})$ has a refinement $\mathcal{V}_1 \in P(\mathcal{U})$ (see Notations 1). If for every $\mathcal{U}, \mathcal{V} \in \mathcal{B}(X)$, $\overline{\mathcal{V}} \subset \mathcal{U}$

compact, the map $\varphi_{\mathcal{U}\mathcal{V}}$ is continuous (which is true if μ is the collection), one can easily see that $\tau_{\mathcal{U}\mathcal{V}} \leq$ $\leq \tau_{\mathcal{U}\mathcal{V}_1}$ (or see [2], 1.1.15). Thus $\tau^*_{\mathcal{U}} = \varinjlim_{\mathcal{V} \in P(\mathcal{U})} \tau_{\mathcal{U}\mathcal{V}}$. For every $\mathcal{V} \in P(\mathcal{U})$ we have $\tau_{\mathcal{U}\mathcal{V}_0(\mathcal{U})} \leq \tau_{\mathcal{U}\mathcal{V}}$, because every such $\mathcal{V}$ refines $\mathcal{V}_0(\mathcal{U})$. However, if $\mathcal{S}$ moreover satisfies (5) (i.e. if μ is finitely projective whenever μ is a collection), we get by Theorem 10 conversely $\tau_{\mathcal{U}\mathcal{V}} \leq \tau_{\mathcal{U}\mathcal{V}_0(\mathcal{U})}$, because by (9),(10): $\tau_{\mathcal{U}\mathcal{V}} = \varprojlim_{V \in \mathcal{V}} \tau_V \leq \varprojlim_{V \in \mathcal{V}} \tau'_V = \tau'_{\mathcal{U}} = \tau^*_{\mathcal{U}} = \tau_{\mathcal{U}\mathcal{V}_0(\mathcal{U})}$. Thus $\tau^*_{\mathcal{U}} = \tau_{\mathcal{U}\mathcal{V}}$ for every $\mathcal{V} \in P(\mathcal{U})$, which gives Theorem A and moreover Theorem 1.1.37 in [2], too.

13. Remark. Z.Frolík introduced the notions of topologized presheaf and of presheaf of topological spaces (see [1],p.59). Let $\mathcal{S}$ from (2) be a presheaf of sets with closures described in (2), $\mu = \{\tau_{\mathcal{U}}\}$ its set of closures. If each $\tau_{\mathcal{U}}$ is formed by the topology $u_{\mathcal{U}}$, we say that $\mathcal{S} = \{(S_{\mathcal{U}}\, u_{\mathcal{U}}); \varphi_{\mathcal{U}V}; X\}$ is a topologized presheaf and the collection $\mu = \{u_{\mathcal{U}}\}$ is called the topologization of $\mathcal{S}$. If moreover each $\varphi_{\mathcal{U}V} : (S_{\mathcal{U}}\, u_{\mathcal{U}}) \longrightarrow (S_V\, u_V)$ is continuous, we say, $\mathcal{S}$ is a presheaf of topological spaces. The topologization μ is then called compatible topologization.

The definitions, notions, theorems and the methods of proofs emplyed in this paper would not lose their sense and validity also in the case if we worked especially in topologies, i.e. if we considered only topologized presheafes instead of presheaves of sets with closures. The reformulation of the results for this case is very easy and the proofs are quite analogous. We will study this case briefly.

Definition D. Let $\mathcal{S} = \{(S_U \ \mu_U) ; \rho_{UV} ; X\}$ be a topologized presheaf, $\mu = \{\mu_U\}$ its topologization. Following (4) we define for $U \in \mathcal{B}(X), \mathcal{V} \in \Pi(U)$ the closure $\tau_{U\mathcal{V}}$ in S_U by the maps $\rho_{UV} : S_U \to \to (S_V \ \mu_V)$ projectively. Then $\tau_{U\mathcal{V}}$ is a topology $\mu_{U\mathcal{V}}$. We say μ is projective, if $\mu_U = \mu_{U\mathcal{V}}$ for every $U, \mathcal{V}$.

A topology-projective modification of μ is the finest projective topologization $\mu' = \{\mu'_U\}$ of $\mathcal{S}$ coarser than μ.

A closure-projective modification of μ is the finest projective closure-collection $\mu'' = \{\tau''_U\}$ of $\mathcal{S}$ coarser than μ.

For a topologization μ of $\mathcal{S}$ there exist the topology-projective modification μ' by [1] - p.59 and the closure-projective modification μ'' by [2] - p.116. Obviously $\mu \leq \mu'' \leq \mu'$.

Since the assumption (5) is not changed in the case of a topologized presheaf and since $\tau_{U\mathcal{V}} = \mu_{U\mathcal{V}}$ is a topology for every $U, \mathcal{V}$, Theorem 7 asserts:

Theorem 7′. Let $\mathcal{S} = \{(S_U \ \tau_U); \rho_{UV} ; X\}$ be a topologized presheaf over a locally compact space X, $\mu = \{\mu_U\}$ its topologization. Let μ satisfy the assumption (5). Then $\mu^{\#} = \{\mu^{\#}_U\} = \{\mu_{U\mathcal{V}_0(U)} ; U \in \mathcal{B}(X)\}$ is a projective topologization of $\mathcal{S}$.

The proof and Corollary 8 are the same, if we write "topologization", resp. "compatible topologization" instead of "set of closures", resp. "closure collection". Theorem 10 in a topological case asserts:

Theorem 10'. Let $\mathcal{S}$ be a topologized presheaf over a locally compact space X, $\mu = \{\mu_{\mathcal{U}}\}$ its topologization which satisfies the assumption (5). Suppose for every $\mathcal{U}, V$ with the property $\overline{V} \subset \mathcal{U}$ is compact, the map $\rho_{\mathcal{U}V}$ is continuous. Then $\mu^{\#} = \mu'' = \mu'$ (see Definition D).

Proof. The continuity of $\rho_{\mathcal{U}V}$ for $\overline{V} \subset \mathcal{U}$ compact implies $\mu \leq \mu^{\#} = \{\mu_{\mathcal{U}\, \mathcal{V}_0(\mathcal{U})}\}$. Let $\mu'' = \{\tau''_{\mathcal{U}}\}$. Since $\mu_{\mathcal{U}} \leq \tau''_{\mathcal{U}}$ for all $\mathcal{U} \in \mathcal{B}(X)$ and since μ'' is a projective collection, we get $\mu^{\#}_{\mathcal{U}} = \varprojlim_{V \in \mathcal{V}(\mathcal{U})} \mu_V \leq \varprojlim_{V \in \mathcal{V}_0(\mathcal{U})} \tau''_V = \mu''_{\mathcal{U}}$. By Theorem 7' $\mu^{\#}$ is a projective topologization and therefore by Definition D $\mu^{\#} = \mu'' = \mu'$.

Theorem 10' shows that in the case of a topologization μ of $\mathcal{S}$ the closure-projective modification μ'' of μ is again a topologization of $\mathcal{S}$ and therefore it coincides with μ'.

Following (12) we denote by $\mu^{*}_{\mathcal{U}} = \varinjlim_{V \in \pi(\mathcal{U})} \mu_{\mathcal{U}V}$ the finest topology in $S_{\mathcal{U}}$ coarser than every $\mu_{\mathcal{U}V}$. Then Remark 11 is not changed if we write the topology $\mu^{*}_{\mathcal{U}}$ instead of the closure $\tau^{*}_{\mathcal{U}}$.

R e f e r e n c e s :

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Matematicko-fyzikální fakulta
Karlova universita
Praha 8, Sokolovská 83, Československo