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Josef Daneš

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Josef DANEŠ, Praha

§ 0. Introduction. This paper is devoted to the study of fixed points of some mappings in metric and normed spaces. Notations and terminology are described in Section 1. Section 2 contains some results near to those given by Kannan in [11] and Kirk in [13]. In Section 3 we study k-mcL mappings and the relation between Fréchet differentiability and the measure of non-compactness. Section 4 is devoted to an application of a theorem of Browder [4].

§ 1. Notations and terminology. Let (X, d) and (Y, e) be two pseudometric spaces, C a subset of X and T a mapping of X into Y. Then T is said to be uniformly continuous on C with respect to X, if for each positive δ there is a positive ε such that if c is in C and x in X with $d(c, x) \leqq \varepsilon$, then $e(T(c), T(x)) \leqq \delta$.

Let M be a subset of X and define

$Q(M) = \{\varepsilon > 0 : M$ can be covered by a finite number of closed ε-balls in $X\}$

and the measure of non-compactness of the set M by $\chi(M) = \inf Q(M)$ (see Sadovskii [14]). For elementary properties of the measure of non-compactness and related topics

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see [3],[8],[9],[15]. T is called a k-mcL mapping if $\chi(T(M)) \leq k\,\chi(M)$ for any subset M of X. T is called a strictly k-mcL mapping [1] if $\chi(T(M)) < k\,\chi(M)$ for any non-precompact bounded subset M of X. In this terminology, T is concentrative if it is continuous and a strictly 1-mcL mapping. T is asymptotically regular (see [5]), if $e(T^{n}(x), T^{n+1}(x)) \longrightarrow 0$ as $n \longrightarrow +\infty$, for any x in X. It is easy to see that T is uniformly continuous on C with respect to X, respectively a k-mcL mapping, if it is k-Lipschitzian on C with respect to X (that is c in C and x in X implies that $e(T(c), T(x)) \leq k\cdot d(c,x)$ for some $k \geq 0$), respectively k-Lipschitzian on X.

Let (X,p) and (Y,q) be pseudonormed linear spaces and X_1 and Y_1 their closed unit balls at the origin. In what follows, "$\rightharpoonup$" and "$\longrightarrow$" denote the convergence in the weak and strong (pseudonorm) topology, respectively. In [8] and [10] we computed the measure of non-compactness of X_1: $\chi(X_1) = 0$ or 1 if $X/p^{-1}(0)$ has a finite or infinite dimention. If T is a linear mapping of X into Y, denote by $\chi(T)$ the number $\chi(T(X_1))$. It is easy to see that χ is a pseudonorm on the space of all linear bounded mappings from X into Y; its kernel, that is the set $\chi^{-1}(0)$, consists of precompact linear mappings of X into Y. Clearly, $\chi(T) \leq \|T\|$ for any linear $T: X \longrightarrow Y$.

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1) k-mcL is the abbreviation of "Lipschitzian in the sense of the measure of non-compactness with constant k".

Now, let X and Y be normed linear spaces, C a subset of X and T a mapping of C into Y. Then T is said to be (a) <u>demicontinuous</u> if $x_n \longrightarrow x_0$ in C implies $T(x_n) \rightharpoonup T(x_0)$ in Y; (b) <u>weakly continuous</u> if $x_n \rightharpoonup x_0$ in C implies $T(x_n) \rightharpoonup T(x_0)$ in Y; (c) <u>convex</u> if the functional $f(x) = \|x - T(x)\|$ and the set C are convex; (d) <u>Fréchet differentiable at a point x in C</u> (see [16]) if x is in the interior of C and $T(x+h) = T(x) + T'(x)h + \omega(x,h)$ $(h \in X \cap (C - x))$ [2], where $T'(x)$, the Fréchet derivative of T at x, is a linear continuous mapping of X into Y and $\omega(x,h)$, the remainder of T at x, satisfies the condition: $\lim_{h\to 0} \dfrac{\|\omega(x,h)\|}{\|h\|} = 0$;
(e) <u>uniformly Fréchet differentiable on</u> C (see [16]) if C is open, T is Fréchet differentiable at any x in C and $\lim_{h\to 0} \dfrac{\|\omega(x,h)\|}{\|h\|} = 0$ uniformly for x in C; (f) <u>feebly semicontractive</u> if $Y = X =$ a Banach space and there is a mapping V of $C \times C$ into X such that $T(x) = V(x,x)$ for all x in C, $\|V(x,z) - V(y,z)\| \leq \|x - y\|$ $(x, y, z$ in $C)$ and the map $x \longrightarrow V(\cdot, x)$ is compact from C to the space of maps of C to X with the uniform metric. The kernel of C is the set $K(C) = \{x \in X:$ C is starshaped with respect to x, that is, the closed segment $[x, z]$ is contained in C for any z in $C\}$.

§ 2. In this section we shall present some sufficient conditions on the existence of fixed points of some mappings in metric spaces. These results are related to those of

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2) $C - x$ denotes the set $\{c - x : c \in C\}$.

Kannan [11] and Kirk [13].

__Theorem 1__. Let (X, τ) be a non-empty compact space and d a non-negative real-valaed symmetric function on $X \times X$ such that $d(x, y) = 0$ implies $x = y$ $(x, y \in X)$. Suppose that T_1 and T_2 are mappings of X into itself satisfying the following conditions:

(1) if $T_1(x) = x = y = T_2(y)$ is not true, then
$$d(T_1(x), T_2(x)) < \frac{1}{2}[d(x, T_1(x)) + d(y, T_2(y))] \ ;$$

(2) the function $f(x, y) = d(x, T_1(x)) + d(y, T_2(y))$ is lower semi-continuous on $(X, \tau) \times (X, \tau)$.

Then the mappings T_1 and T_2 have a common fixed point which is the unique fixed point of each of T_1 and T_2.

__Proof__. If z and w are fixed points of T_1 and T_2 respectively, with $z \neq w$, then by (1) we have $d(T_1(z), T_2(w)) < \frac{1}{2}[0+0] = 0$, a contradiction, proving the trivial part of the theorem.

Since $f(x, y)$ is a lower semi-continuous function on the (non-empty) compact space $(X, \tau) \times (X, \tau)$, there is a point (z, w) in $X \times X$ at which f attains its infimum. If

$$(*) \qquad T_1(T_2(w)) = T_2(w) = w$$

or

$$(**) \qquad z = T_1(z) = T_2(T_1(z))$$

is true, then w or z is a common fixed point of T_1 and T_2. Hence it suffices to prove that at least one of $(*)$ and $(**)$ is satisfied. Suppose not. Then, by (1)

$$f(T_2(w), T_1(z)) = d(T_2(w), T_1(T_2(w))) + d(T_1(z), T_2(T_1(z))) =$$
$$= d(T_1(T_2(w)), T_2(w)) + d(T_1(z), T_2(T_1(z))) <$$

$$< \frac{1}{2}[d(T_2(w), T_1(T_2(w))) + d(w, T_2(w))] +$$
$$+ \frac{1}{2}[d(x, T_1(x)) + d(T_1(x), T_2(T_1(x)))] =$$
$$= \frac{1}{2}[f(x, w) + f(T_2(w), T_1(x))] ,$$

that is, $f(T_2(w), T_1(x)) < f(x, w)$ – a contradiction to the minimality of f at the point (x, w).

In the above theorem one can take, for instance, as d a metric on X. Proofs of the following corollaries are similar to those given in [7],[10]. We can obtain further assertions by taking $T_1 = T_2 = T$.

Corollary 1. Let (X, τ) be a non-empty compact space and d a non-negative real-valued lower semi-continuous function on $(X, \tau) \times (X, \tau)$. Suppose that T_1 and T_2 are continuous mappings of X into itself satisfying the condition (1) of Theorem 1. Then the conclusion of Theorem 1 remains valid.

Corollary 2. Let X be a non-empty weakly compact subset of a normed linear space, T_1 and T_2 weakly continuous mappings of X into itself satisfying the condition (1) of Theorem 1 with $d(x, y) = \| x - y \|$. Then the conclusion of Theorem 1 remains valid.

Corollary 3. Let X be a non-empty weakly compact convex subset of a normed linear space, T_1 and T_2 demicontinuous mappings of X into itself satisfying the condition (1) of Theorem 1 with $d(x, y) = \|x - y\|$. Let the function f (see Theorem 1) be convex on $X \times X$. Then the conlusion of Theorem 1 remains valid.

Corollary 4. Let X, T_1, T_2 and d be as in Corollary

3. Suppose that $I - T_1$ and $I - T_2$ are convex. (I denotes the identity mapping on X .) Then the conclusion of Theorem 1 remains valid.

Theorem 2. Let (X,d) be a complete metric space, C a non-empty compact subset of X and T a (not necessarily continuous) mapping of X into itself which is uniformly continuous on C with respect to X. Let $\alpha(T,x)$ be a subset of X, for any $x \in X$. Suppose that:

(1) $\inf_{x \in X} d(x, T(x)) = 0$;

(2) $\overline{\alpha(T,x)} \cap C \neq \emptyset$ for each x in X ;

(3) $d(y, T(y)) \leq \kappa(d(x, T(x)))$ for each $y \in \alpha(T,x)$, $x \in X$, where $\kappa(t)$ is a function defined on $(0, +\infty)$ with $\kappa(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0+$.

Then T has a fixed point in X (even in C).

Proof. Let $\varepsilon > 0$ be given. Then, by (1), there exists a point x in X such that $d(x, T(x)) < \varepsilon$; by (2), there are y in $\alpha(T,x)$ and c in C with $d(y,c) < \varepsilon$. Thus, by (3), we have

$$d(c, T(c)) \leq d(c,y) + d(y, T(y)) + d(T(y), T(c)) \leq$$

$$\leq \varepsilon + \kappa(\varepsilon) + \delta(\varepsilon) = \eta(\varepsilon) \ ,$$

where $\delta(\varepsilon) = \sup\{d(T(z), T(w)) : z \in X, w \in C, d(z,w) \leq \varepsilon\}$ is the modul of uniform continuity of T on C with respect to X. The fact $\eta(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0+$ implies that $\inf_{c \in C} d(c, T(c)) = 0$. The continuity of T on the non-empty compact subset C ensures the existence of a point x_0 in C such that $d(x_0, T(x_0)) = \inf_{c \in C} d(c, T(c)) = 0$, and x_0 is a fixed point of T .

Remark. The condition (1) of Theorem 2 is satisfied if (X,d) is a bounded complete subset of a normed linear space and T is a nonexpansive mapping of X into itself and the kernel of X intersects the range of T, $K(X)\cap R(T)\neq\emptyset$ (see [10],Proposition 4), or if T is asymptotically regular, $d(T^n(x), T^{n+1}(x)) \longrightarrow 0$ as $n \longrightarrow +\infty$, for any x in X. In many cases we can take $\alpha(T,x)\subset\{T^n(x)\}_{n=0}^{\infty}$, or $\alpha(T,x)\subset \mathrm{co}\{T^n(x): n=0,1,\dots\}$, if X is a subset of a linear space (cf. Kirk [13],Cor.2.1).

§ 3. k-mcL mappings and Fréchet differentiable mappings.

Proposition 1. Let (X,p) and (Y,q) be pseudonormed linear spaces and T a linear mapping of X into Y. Then:

(1) T is continuous if and only if $\chi(T) < +\infty$;

(2) T is precompact (that is, it maps bounded subsets of X onto precompact subsets of Y) if and only if $\chi(T)=0$;

(3) if T is continuous then it is a $\chi(T)$-$m c$L mapping;

(4) if T is not precompact, then T is not a k-mcL mapping for any $k<\chi(T)$.

Proof. (1) and (2) follow at once from the definition of $\chi(T)$ and Lemma 1, (2) and (3) in [9]. The same considerations as in the proof of Theorem 8 in [10] prove (3). The part (4) of the theorem is a consequence of the equality $\chi(T)\equiv\chi(T(X_1))=\chi(T)\cdot\chi(X_1)$. (Note that $\chi(T)>0$ implies that the dimension of the quotient space $X/p^{-1}(0)$ is infinite and $\chi(X_1)=1$, cf. Proposition 6 in [10].)

Proposition 2. Let (X,d) and (Y,e) be pseudometric spaces and $\{T_n\}_{n=1}^{\infty}$ a sequence of k-mcL mappings of X into Y which converges, uniformly on bounded subsets of X, to a mapping T of X into Y. Then T is a k-mcL mapping.

Proof. Let $\varepsilon > 0$ be given and let M be a bounded subset of X. Then there exists n_0 such that $e(T_{n_0}(x), T(x)) \leqq \varepsilon$ for all x in M. Hence the Hausdorff distance (with respect to e) of $T_{n_0}(M)$ and $T(M)$ is not greater than ε and, using [3], § 3, Lemma, or [8], Theorem 1.11, respectively [9], Lemma 1, (8), we obtain that $|\chi(T_{n_0}(M)) - \chi(T(M))| \leqq \varepsilon$. Hence $\chi(T(M)) \leqq \chi(T_{n_0}(M)) + \varepsilon \leqq k \cdot \chi(M) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $\chi(T(M)) \leqq k\,\chi(M)$.

Theorem 3. Let X and Y be normed linear spaces, C an open non-empty subset of X and T a mapping of C into Y possessing the Fréchet derivative at a point x of C. Then $\lim_{\varepsilon \to 0+} \dfrac{\chi(T(x+\varepsilon X_1))}{\varepsilon}$ exists and equals to $\chi(T'(x))$.

Proof. There is an $\varepsilon_0 > 0$ such that the closed ε_0-ball at x is contained in C. We can write

$$T(x+h) = T(x) + T'(x)h + \omega(x,h) \quad (\|h\| \leqq \varepsilon_0,\ h \in X),$$

where $\delta(\varepsilon) = \sup\{\dfrac{\|\omega(x,h)\|}{\|h\|} : h \in X,\ 0 < \|h\| \leqq \varepsilon\}$ converges to 0 as ε tends to 0. Further,

$$T(x+\varepsilon X_1) \subset T(x) + T'(x)(\varepsilon X_1) + \omega(x, \varepsilon X_1) \quad (0 < \varepsilon \leqq \varepsilon_0),$$

$$T'(x)(\varepsilon X_1) \subset T(x) - T(x+\varepsilon X_1) + \omega(x, \varepsilon X_1),$$

hence

$$\frac{T(x+\varepsilon X_1)}{\varepsilon} \subset \frac{T(x)}{\varepsilon} + T'(x)(X_1) + \frac{\omega(x,\varepsilon X_1)}{\varepsilon}$$

$$T'(x)(X_1) \subset \frac{T(x)}{\varepsilon} - \frac{T(x+\varepsilon X_1)}{\varepsilon} + \frac{\omega(x,\varepsilon X_1)}{\varepsilon} \qquad (0<\varepsilon \leq \varepsilon_0)$$

Thus

$$\frac{T(x+\varepsilon X_1)}{\varepsilon} \subset \frac{T(x)}{\varepsilon} + T'(x)(X_1) + \delta(\varepsilon) X_1$$

$$(0<\varepsilon \leq \varepsilon_0)$$

$$T'(x)(X_1) \subset \frac{T(x)}{\varepsilon} - \frac{T(x+\varepsilon X_1)}{\varepsilon} + \delta(\varepsilon) X_1 ,$$

that is

$$\left| \frac{\chi(T(x+\varepsilon X_1))}{\varepsilon} - \chi(T'(x)) \right| \leq \delta(\varepsilon) \ (0<\varepsilon \leq \varepsilon_0),$$

and the theorem follows.

Remark. A direct consequence of the proof is that if T is uniformly Fréchet differentiable on C , then $\frac{\chi(T(x+\varepsilon X_1)}{\varepsilon}$ converges to $\chi(T'(x))$ as $\varepsilon \to 0$, uniformly for x in C .

Corollary 1. Let X and Y be normed linear spaces, C an open non-empty subset of X and T a mapping of C into Y possessing the Fréchet derivative at a point x in C . If T is a k-mcL mapping, then so is its Fréchet derivative $T'(x)$, that is $\chi(T'(x)) \leq k$.

Proof. The proof is a direct consequence of Theorem 3 and [10], Proposition 6, respectively [8], Theorem 1.7.

Lemma 1. Let X and Y be normed linear spaces, C a non-empty bounded subset of X which is starshaped with respect to the origin of X and T an a-homogeneous mapping of C into Y for some $a \leq 1$ (that is $T(tx) = t^a T(x)$ if $t > 0$ and $x, tx \in C$) and a k-mcL mapping on $C \cap X_1$ for some $k \geq 0$. Then T is a (strictly) k-mcL mapping on C.

Proof. We can restrict our consideration to the case when T is a k-mcL mapping on $C \cap X_1$. Let M be a bounded subset of C and denote $M_1 = M \cap X_1$ and $M_2 = M \cap \cap (X \setminus X_1)$. Then there is a $t > 1$ such that $t^{-1} M_2$ is contained in X_1. Then $\chi(T(M_2)) = \chi(t^a T(t^{-1} M_2)) = = t^a \chi(T(t^{-1} M_2)) \leq t^a \cdot k \cdot \chi(M_2)$. Therefore

$$\chi(T(M)) = \chi(T(M_1) \cup T(M_2)) = \max\{\chi(T(M_1)),$$

$$\chi(T(M_2))\} \leq \max\{k \cdot \chi(M_1), k \cdot \chi(M_2)\} = k \cdot \chi(M).$$

§ 4. <u>An application of a Browder's theorem</u>. Recently, Browder [4] has proved the following important theorem:

Let X be a Banach space, C a closed bounded convex subset of X having the origin of X in its interior, T a mapping of C into X such that for each x in the boundary of C, $Tx \neq \lambda x$ for any $\lambda > 1$. Suppose that for a given constant $k \leq 1$ and a mapping V of $C \times C$ into X, $T(x) = V(x, x)$ for all x in C while

$$\|V(x,z) - V(y,z)\| \leq k \|x - y\| \qquad (x, y \in C)$$

and the map $x \longrightarrow V(\cdot, x)$ is compact from C to the space of maps from C to X with the uniform metric. Then:

(a) If $k < 1$, T has a fixed point in C.

(b) If $k \leq 1$ and $(I - T)(C)$ is closed in X, then T has a fixed point in C.

By means of this theorem, Browder [4] derived a fixed point theorem for semicontractive mappings in uniformly convex Banach spaces, and Kirk [12] made this for strongly semicontractive mappings in reflexive Banach spaces. Our purpose in this section is to give a fixed point theorem for concentrative feebly semicontractive mappings in Banach spaces. In the part (b) of the Browder's theorem, the problem is to prove that $(I-T)(C)$ is closed in X.

Lemma 2. Let X be a normed linear space, C a complete subset of X and T a concentrative mapping of C into X. Then the mapping $I-T$ maps bounded closed subsets of C into bounded closed subsets of X (I denotes the identity mapping of C into C).

Proof. Let M be a closed and bounded subset of X. Since T is concentrative, we have $\chi(T(M)) \leqq \chi(M) < +\infty$ and hence $T(M)$ is bounded. Now, the inclusion $(I-T)(M) \subset$ $\subset M - T(M)$ implies the boundedness of $(I-T)(M)$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $(I-T)(M)$ converging (strongly) to a point y_0 in X. Then there are points x_n in M such that $x_n - T(x_n) = y_n$. Denote $A = \{x_n : n = 1, 2, \dots\}$ and $B = \{y_n : n = 1, 2, \dots\}$. Then, clearly, $A \subset T(A) + B$ and $T(A) \subset A - B$. Thus, B being precompact (the underlying set of a convergent sequence), we have
$\chi(A) \leqq \chi(T(A)) + \chi(B) = \chi(T(A)) \leqq \chi(A) + \chi(B) = \chi(A)$,
that is, $\chi(T(A)) = \chi(A)$, and hence A is precompact. Then $\bar{A}$ is a compact subset of C. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow x_0$ for some x_0 in C. We have $T(x_{n_k}) \longrightarrow T(x_0)$ since T is continuous. Hence $x_0 - T(x_0) = y_0$ and y_0 is in $(I-T)(C)$ which proves the lemma.

Lemma 3. Let X be a normed linear space, C a complete subset of X and T a concentrative mapping of C into X.

If $x_n \longrightarrow x_0$ and $y_n = x_n - T(x_n) \longrightarrow y_0$ for some $\{x_n\} \subset C$, $x_0 \in C$ and $y_0 \in X$, then $y_0 = x_0 - T(x_0)$.

Proof. Denoting $A = \{x_n\}$ and $B = \{y_n\}$ and using $A \subset T(A) + B$, $T(A) \subset A - B$, we have, by the same argument as in the proof of the preceding lemma, $\chi(A) = 0$. Hence $\bar{A}$ is compact and $x_n \longrightarrow x_0$ in $\bar{A}$ implies $x_n \longrightarrow x_0$. Therefore, $y_0 = x_0 - T(x_0)$.

Theorem 4. Let X be a Banach space, C a closed bounded convex subset of X having the origin of X in its interior, T a concentrative feebly semicontractive mapping of C into X satisfying the Leray-Schauder condition: for each x in the boundary of C and for each $\lambda > 1$, $Tx \neq \lambda x$. Then T has a fixed point in C.

Proof. By Lemma 2, $(I - T)(C)$ is closed, and using the Browder's theorem mentioned at the beginning of this section, our theorem follows.

Corollary 1. Let X and C be as in the theorem. Let T be a concentrative nonexpansive mapping of C into X satisfying the Leray-Schauder condition (see Theorem 4). Then T has a fixed point in C.

Corollary 2. Let X and C be as in the theorem. Let T be the sum of a concentrative nonexpansive mapping and a compact mapping of C into X. Suppose that T satisfies the Leray-Schauder condition (see Theorem 4). Then T has a fixed point in C.

Lemma 4. Let X be a normed linear space and $\{x_n\}$ a sequence in X weakly converging to x_0 and let ε be a real number greater than $\chi(\{x_n : n = 1, 2, \ldots\})$. Then there is n_0 such that for each $n \geq n_0$ x_n lies in the

2ε -ball at x_0 .

<u>Proof</u>. Suppose not. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is disjoint from the 2ε -ball at x_0. Now, $\{x_n\}$, and hence $\{x_{n_k}\}$, is covered by a finite number of closed ε -balls. Hence there exist a point z in X and a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ contained in the closed ε -ball at z . Since the closed ε -ball at z is convex and $x_{n_{k_j}} \longrightarrow x_0$,the point x_0 lies in the closed ε -ball at z . Thus, $\{x_{n_{k_j}}\}$ being contained in the closed ε -ball at z , it is contained in the closed 2ε -ball at x_0 , a contradiction.

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Matematicko-fyzikální fakulta
Karlova Universita
Sokolovská 83, Praha 8
Československo

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